Chapter 1

Differential Topology

1.1 Tensors

A **tensor** is a mathematical object that directly represents a physical quantity. Tensors of the same type can be added, and multiplied by a scalar, in the usual way. Scalars and vectors are tensors, but many physical quantities are some other type of tensor.

A **scalar** is a tensor that behaves like a number. Examples of spatial[1] scalars are time t, energy E and electric potential ϕ. Examples of spacetime scalars are proper time τ, mass m and charge q.

1.1.1 Vectors and covectors

A **vector** is a tensor that behaves like an arrow. Their properties inspire the vector

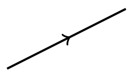

Figure 1.1.1: A vector.

space axioms of mathematics. A scalar times a vector is a vector and the sum of two vectors is a vector, see Figure 1.1.2. Examples of vectors are displacement $\vec{dx}$, velocity

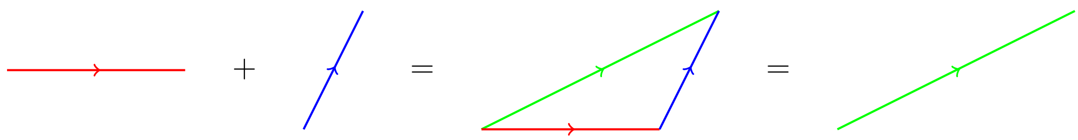

Figure 1.1.2: The sum of two vectors is a vector.

[1]Physical quantities may be one type of tensor with respect to one space but another type of tensor with respect to another space. For example, a displacement in time is a scalar with respect to space but a vector with respect to time. Unless otherwise specified, the space can be assumed to be space, or spacetime in the context of relativity.

$$\vec{v} \equiv \frac{d\vec{x}}{dt} \tag{1.1.1}$$

and acceleration

$$\vec{a} \equiv \frac{d\vec{v}}{dt} \tag{1.1.2}$$

A **covector** or **one-form** is a tensor that behaves like the local linearized form of contour lines or the crests of a wave, see Figure 1.1.3. A scalar times a covector

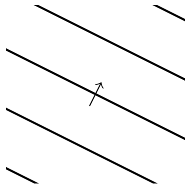

Figure 1.1.3: A covector.

is a covector and the sum of two covectors is a covector, see Figure 1.1.4. Examples

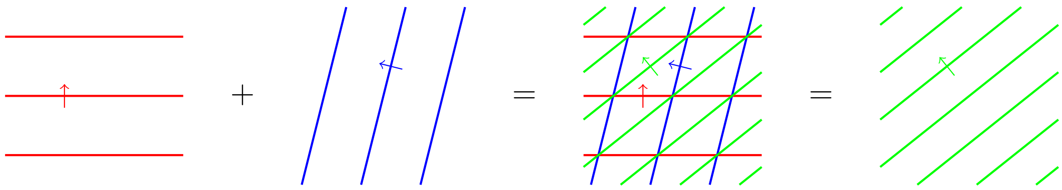

Figure 1.1.4: The sum of two covectors is a covector.

of covectors are the gradient of a scalar field $\underline{\nabla}\phi$, wave "vectors" $\underline{k}$, electric field and magnetic "vector" potential

$$\underline{E} = -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t} \tag{1.1.3}$$

momentum

$$\underline{p} = \hbar \underline{k} \tag{1.1.4}$$

and force

$$\underline{F} = q\underline{E} \tag{1.1.5}$$

or

$$\underline{F} = \frac{d\underline{p}}{dt} \tag{1.1.6}$$

A vector can be **contracted** with a covector to give a scalar

$$\vec{v} \cdot \underline{\omega} = \text{scalar} \tag{1.1.7}$$

corresponding to the number of covector planes crossed by the vector, with sign given by the relative orientations of the vector and covector, see Figure 1.1.5. For example, a

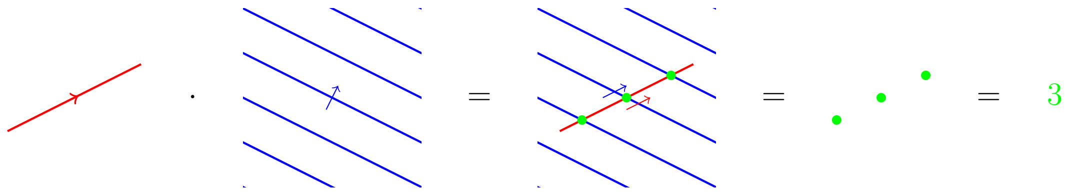

Figure 1.1.5: A vector contracted with a covector gives a scalar.

displacement contracted with the gradient of a scalar field gives the change in the scalar field

$$\vec{dx} \cdot \underline{\nabla}\phi = d\phi \tag{1.1.8}$$

and power equals force contracted with velocity

$$P = \underline{F} \cdot \vec{v} \tag{1.1.9}$$

Comparing vectors and covectors, the magnitude of a vector is given by its length, while the magnitude of a covector is given by the density of its planes. The direction of a vector is along its length (intrinsically oriented), while the direction of a covector is normal to its planes (extrinsically oriented) in the sense that $\underline{n} \cdot \vec{v} = 0$ for any vector $\vec{v}$ lying in the plane of the covector $\underline{n}$. Thus, a vector is an intrinsically oriented dimension one plane element, while a covector is an extrinsically oriented codimension[2] one plane density.

1.1.2 Exterior algebra

Vectors or covectors can be multiplied together using the **exterior** or **wedge product**, generating **multivectors** or **differential forms** respectively. Multivectors, differential forms and their exterior algebra have an elegant mathematical structure and clear physical interpretation.

The exterior product of two vectors is the **two-vector** given by the oriented plane element formed by the two vectors, see Figure 1.1.6. Note that the shape of the plane

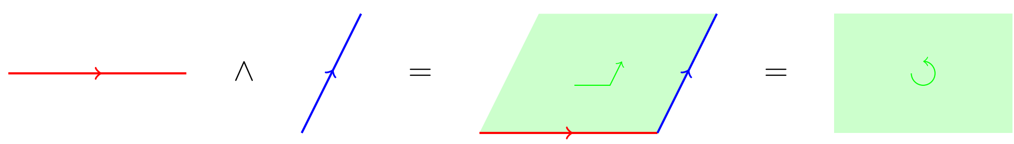

Figure 1.1.6: The exterior product of two vectors is a two-vector.

element does not matter

$$(2\vec{a}) \wedge \vec{b} = \vec{a} \wedge (2\vec{b}) = 2(\vec{a} \wedge \vec{b}) \tag{1.1.10}$$

[2]Codimension d is dimension $D - d$ where D is the dimension of the space.

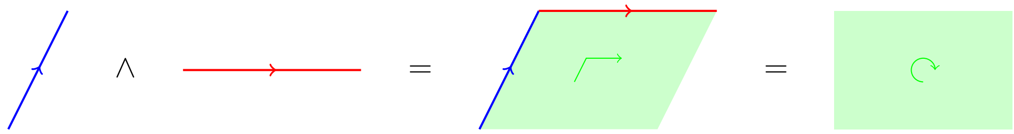

Figure 1.1.7: The exterior product is antisymmetric.

only its plane, area and orientation. The exterior product is antisymmetric

$$\vec{a} \wedge \vec{b} = -\vec{b} \wedge \vec{a} \tag{1.1.11}$$

since swapping the vectors in Figure 1.1.6 would reverse the orientation, see Figure 1.1.7.

A scalar times a two-vector is a two-vector and the sum of two two-vectors is a

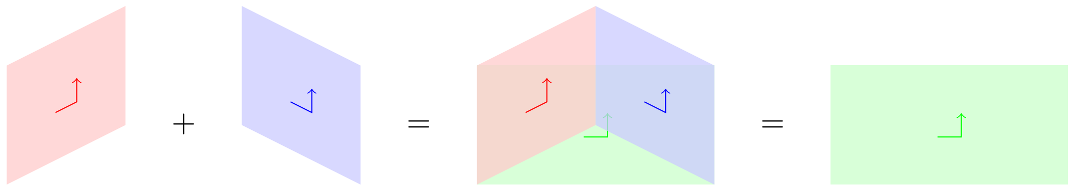

Figure 1.1.8: The sum of two two-vectors is a two-vector.

two-vector, see Figure 1.1.8. Examples of two-vectors are surface element $d\vec{\vec{S}}$, angular momentum[3]

$$\vec{\vec{L}} = m\,\vec{x} \wedge \vec{v} \tag{1.1.12}$$

and torque

$$\vec{\vec{\tau}} = \frac{d\vec{\vec{L}}}{dt} \tag{1.1.13}$$

The exterior product of two one-forms is a **two-form** given by the oriented intersections of the one-form planes, see Figure 1.1.9. Note that the position of the intersections

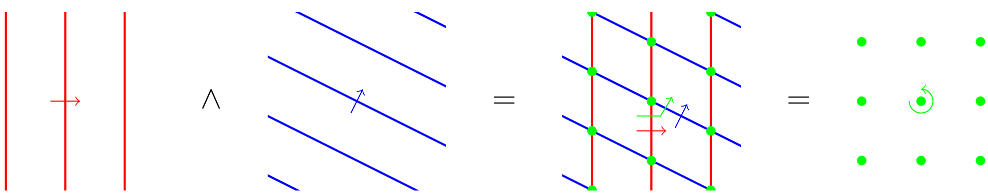

Figure 1.1.9: The exterior product of two one-forms is a two-form.

does not matter, only their density and orientation. A scalar times a two-form is a two-form and the sum of two two-forms is a two-form too, see Figure 1.1.10. Examples of

[3]Note that $\vec{x}$ is a vector, and hence $\vec{\vec{L}}$ is a two-vector, only in flat space. We will onsider more general spaces in Section 1.2.

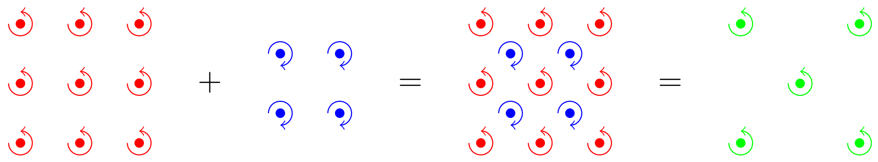

Figure 1.1.10: The sum of two two-forms is a two-form.

two-forms are magnetic flux density

$$\underline{\underline{B}} = \underline{\nabla} \wedge \underline{A} \tag{1.1.14}$$

and electric current density

$$\underline{\underline{j}} = \underline{\underline{\underline{\rho}}} \cdot \vec{v} \tag{1.1.15}$$

A two-form can be contracted with a two-vector to give a scalar,

$$\underline{\underline{\omega}} \cdot \vec{\vec{v}} = \text{scalar} \tag{1.1.16}$$

see Figure 1.1.11. For example, an electric current density contracted with an area

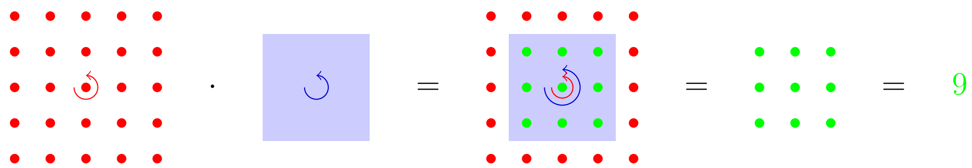

Figure 1.1.11: A two-form contracted with a two-vector gives a scalar.

element gives the current

$$\underline{\underline{j}} \cdot d\vec{\vec{S}} = dI \tag{1.1.17}$$

In general, an n-vector is an intrinsically oriented dimension n plane element, and an n-form is an extrinsically oriented codimension n plane density. The wedge product of an m-form with an n-form is an $(m+n)$-form, and similarly for multivectors. An m-form contracted with an n-vector is an $(m-n)$-form or an $(n-m)$-vector.

The dimension of the space of n-forms is $N!/[n!(N-n)!]$, where N is the dimension of the space, and $0 \leq n \leq N$ due to the antisymmetry of the exterior product. For example, in three dimensions, there are $1, 3, 3, 1$ independent $0, 1, 2, 3$-forms, respectively, and no higher forms. The same is true for multivectors. The multivectors and differential forms in three dimensions are shown in Figure 1.1.12.

Exterior product is associative

$$\boldsymbol{\omega} \wedge (\boldsymbol{\sigma} \wedge \boldsymbol{\rho}) = (\boldsymbol{\omega} \wedge \boldsymbol{\sigma}) \wedge \boldsymbol{\rho} \tag{1.1.18}$$

but contraction is generally not associative

$$\boldsymbol{u} \cdot (\boldsymbol{\omega} \cdot \boldsymbol{v}) = (\boldsymbol{u} \cdot \boldsymbol{\omega}) \cdot \boldsymbol{v} \quad \text{for } \deg \boldsymbol{u} + \deg \boldsymbol{v} \leq \deg \boldsymbol{\omega} \tag{1.1.19}$$

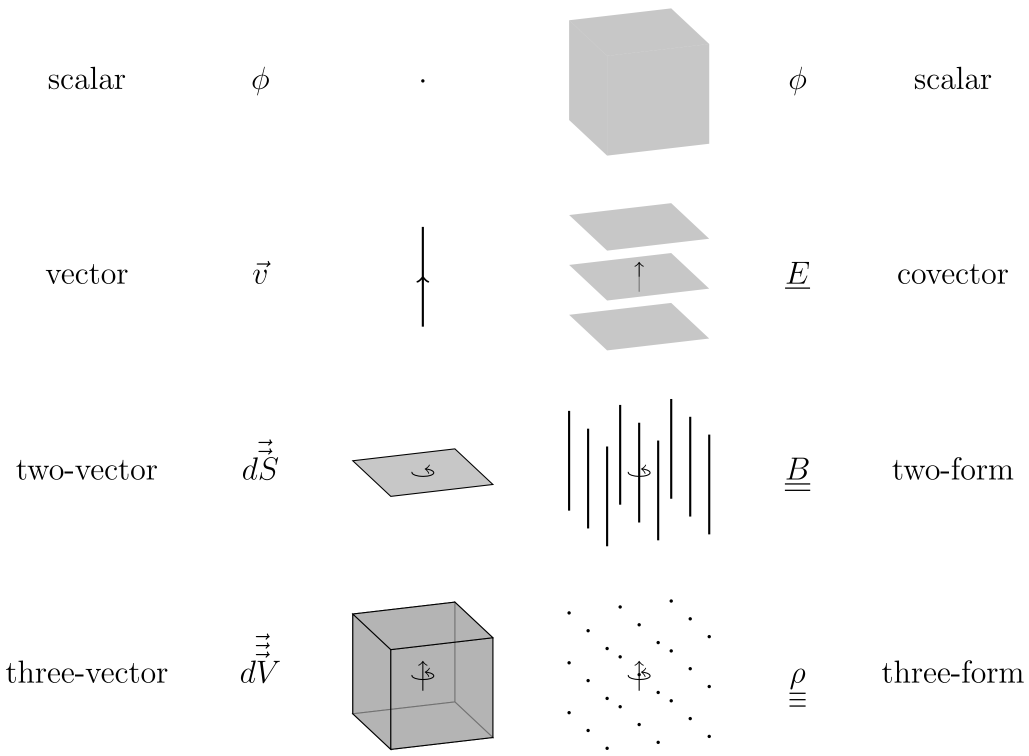

Figure 1.1.12: Multivectors and differential forms in three dimensions.

The antisymmetry of the exterior product gives, for an m-form $\boldsymbol{\omega}$ and an l-form $\boldsymbol{\sigma}$,

$$\boldsymbol{\omega} \wedge \boldsymbol{\sigma} = (-1)^{ml} \boldsymbol{\sigma} \wedge \boldsymbol{\omega} \tag{1.1.20}$$

and similarly for multivectors. Also, for an m-form $\boldsymbol{\omega}$ and an n-vector $\boldsymbol{v}$,

$$\boldsymbol{\omega} \cdot \boldsymbol{v} = \begin{cases} (-1)^{(m-n)n}\, \boldsymbol{v} \cdot \boldsymbol{\omega} & \text{for } n \leq m \\ (-1)^{(n-m)m}\, \boldsymbol{v} \cdot \boldsymbol{\omega} & \text{for } n \geq m \end{cases} \tag{1.1.21}$$

For n-vector $\boldsymbol{v}$, m-form $\boldsymbol{\omega}$ and l-form $\boldsymbol{\sigma}$

$$\boldsymbol{v} \cdot (\boldsymbol{\omega} \wedge \boldsymbol{\sigma}) = \begin{cases} (\vec{v} \cdot \boldsymbol{\omega}) \wedge \boldsymbol{\sigma} + (-1)^{ml}\, (\vec{v} \cdot \boldsymbol{\sigma}) \wedge \boldsymbol{\omega} & \text{for } 1 = n \leq m, l \\ (\underline{\omega} \cdot \boldsymbol{v}) \cdot \boldsymbol{\sigma} + (-1)^{l}\, (\boldsymbol{v} \cdot \boldsymbol{\sigma}) \wedge \underline{\omega} & \text{for } 1 = m \leq n \leq l \\ (\boldsymbol{v} \cdot \boldsymbol{\omega}) \wedge \underline{\sigma} + (-1)^{m}\, (\underline{\sigma} \cdot \boldsymbol{v}) \cdot \boldsymbol{\omega} & \text{for } 1 = l \leq n \leq m \\ (\boldsymbol{\omega} \cdot \boldsymbol{v}) \cdot \boldsymbol{\sigma} + (-1)^{ml}\, (\boldsymbol{\sigma} \cdot \boldsymbol{v}) \cdot \boldsymbol{\omega} & \text{for } l + m - 1 = n \\ (\boldsymbol{v} \cdot \boldsymbol{\sigma}) \cdot \boldsymbol{\omega} & \text{for } l + m \leq n \end{cases} \tag{1.1.22}$$

If $1 < l, m, n < l + m - 1$ then one gets partial contractions of $\boldsymbol{v}$ with both $\boldsymbol{\omega}$ and $\boldsymbol{\sigma}$, for example

$$\vec{\vec{v}} \cdot \left(\underline{\underline{\omega}} \wedge \underline{\underline{\sigma}}\right) = \left(\vec{\vec{v}} \cdot \underline{\underline{\omega}}\right) \underline{\underline{\sigma}} + \underline{\underline{\omega}} \cdot \vec{\vec{v}} \cdot \underline{\underline{\sigma}} + \left(\vec{\vec{v}} \cdot \underline{\underline{\sigma}}\right) \underline{\underline{\omega}} \tag{1.1.23}$$