Homework 12 - Calculus of variations

Q12.1. The action functional

$$S[x(t)] = \int_{t_\mathrm{i}}^{t_\mathrm{f}} L(x, \dot{x}, t)\, dt \tag{Q12.1.1}$$

can be varied either covariantly

$$\frac{\delta S}{\delta x^\mathbf{a}} = \frac{\partial L}{\partial x^\mathbf{a}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\mathbf{a}}\right) \tag{Q12.1.2}$$

or with respect to the coordinate paths

$$\frac{\delta S}{\delta x^\alpha} = \frac{\partial L}{\partial x^\alpha} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) \tag{Q12.1.3}$$

(a) Show that

$$\frac{\delta S}{\delta x^\mathbf{a}} = \frac{\delta S}{\delta x^\alpha} e^\alpha_\mathbf{a} \tag{Q12.1.4}$$

(b) Evaluate Eqs. (Q12.1.2) and (Q12.1.3) for

$$L = \frac{1}{2} m g_\mathbf{ab} \dot{x}^\mathbf{a} \dot{x}^\mathbf{b} - V(x) \tag{Q12.1.5}$$

and show that they are equivalent.

Q12.2. The action functional for a scalar field can be expressed either geometrically

$$S[\phi(x)] = \int L(\phi, \nabla\phi, x)\, \boldsymbol{\epsilon} \tag{Q12.2.1}$$

giving the geometric Euler-Lagrange equation

$$\nabla_\mathbf{a}\left[\frac{\partial L}{\partial(\nabla_\mathbf{a}\phi)}\right] - \frac{\partial L}{\partial \phi} = 0 \tag{Q12.2.2}$$

or in terms of coordinates

$$S[\phi(x)] = \int L(\phi, \partial\phi, x)\, \sqrt{|g|}\, d^4x \tag{Q12.2.3}$$

giving the coordinate Euler-Lagrange equation

$$\partial_\alpha\left[\frac{\partial \mathcal{L}}{\partial(\partial_\alpha\phi)}\right] - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{Q12.2.4}$$

where the Lagrangian density

$$\mathcal{L} = \sqrt{|g|}\, L \tag{Q12.2.5}$$

(a) Show that Eqs. (Q12.2.2) and (Q12.2.4) are equivalent.

(b) Evaluate Eqs. (Q12.2.2) and (Q12.2.4) for

$$L = \frac{1}{2} g^\mathbf{ab} (\nabla_\mathbf{a}\phi)(\nabla_\mathbf{b}\phi) - V(\phi) \tag{Q12.2.6}$$

and show that the resulting equations are equivalent.