Homework 9 - Metric and volume form

Q9.1. Use abstract index notation to show that

$$\star\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{\epsilon} \quad\Rightarrow\quad \boldsymbol{v} = \boldsymbol{\epsilon}^{-1}\cdot\star\boldsymbol{v} \tag{Q9.1.1}$$

A9.1. Using Eq. (2.1.51), for an n-vector $\boldsymbol{v}$ in an N-dimensional space

$$\begin{aligned}
\boldsymbol{\epsilon}^{-1}\cdot\star\boldsymbol{v} &= \boldsymbol{\epsilon}^{-1}\cdot(\boldsymbol{v}\cdot\boldsymbol{\epsilon}) && \text{(A9.1.1)}\\
&\leftrightarrow \frac{1}{n!(N-n)!}\epsilon^{-1\,[\mathbf{a_1\cdots a_n c_{n+1}\cdots c_N}]} v^{[\mathbf{b_1\cdots b_n}]}\epsilon_{[\mathbf{b_1\cdots b_n c_{n+1}\cdots c_N}]} && \text{(A9.1.2)}\\
&= \frac{1}{n!}v^{[\mathbf{b_1\cdots b_n}]}\delta^{[\mathbf{a_1\cdots a_n}]}_{[\mathbf{b_1\cdots b_n}]} && \text{(A9.1.3)}\\
&= v^{[\mathbf{a_1\cdots a_n}]} && \text{(A9.1.4)}\\
&\leftrightarrow \boldsymbol{v} && \text{(A9.1.5)}
\end{aligned}$$

Q9.2. Show that for an n-form $\boldsymbol{\omega}$ in an N-dimensional space

$$*^{-1}\boldsymbol{\omega} = (-1)^{n(N-n)}\,\mathrm{sgn}(g) * \boldsymbol{\omega} \tag{Q9.2.1}$$

A9.2. Using Eqs (2.1.52), (1.4.5), (2.1.43), (2.1.49) and (1.1.19),

$$\begin{aligned}
*^{-1}\boldsymbol{\omega} &= \diamond\star^{-1}\boldsymbol{\omega} && \text{(A9.2.1)}\\
&\leftrightarrow \frac{1}{n!(N-n)!}g_{[\mathbf{a_{n+1}\cdots a_N}][\mathbf{b_{n+1}\cdots b_N}]}\epsilon^{-1\,[\mathbf{b_{n+1}\cdots b_N c_1\cdots c_n}]}\omega_{[\mathbf{c_1\cdots c_n}]} && \text{(A9.2.2)}\\
&= \frac{1}{(n!)^2}\epsilon^{-1}{}_{[\mathbf{a_{n+1}\cdots a_N b_1\cdots b_n}]}g^{[\mathbf{b_1\cdots b_n}][\mathbf{c_1\cdots c_n}]}\omega_{[\mathbf{c_1\cdots c_n}]} && \text{(A9.2.3)}\\
&= \frac{\mathrm{sgn}(g)}{(n!)^2}\epsilon_{[\mathbf{a_{n+1}\cdots a_N b_1\cdots b_n}]}g^{[\mathbf{b_1\cdots b_n}][\mathbf{c_1\cdots c_n}]}\omega_{[\mathbf{c_1\cdots c_n}]} && \text{(A9.2.4)}\\
&= (-1)^{n(N-n)}\frac{\mathrm{sgn}(g)}{(n!)^2}\epsilon_{[\mathbf{b_1\cdots b_n a_{n+1}\cdots a_N}]}g^{[\mathbf{b_1\cdots b_n}][\mathbf{c_1\cdots c_n}]}\omega_{[\mathbf{c_1\cdots c_n}]} && \text{(A9.2.5)}\\
&\leftrightarrow (-1)^{n(N-n)}\,\mathrm{sgn}(g)\star\diamond\boldsymbol{\omega} && \text{(A9.2.6)}\\
&= (-1)^{n(N-n)}\,\mathrm{sgn}(g) * \boldsymbol{\omega} && \text{(A9.2.7)}
\end{aligned}$$

Q9.3. (a) Express Maxwell's equations in terms of $\underline{E}$ and $\underline{\underline{B}}$.

(b) Use

$$\underline{\underline{G}} = *\underline{\underline{F}} - \underline{\underline{N}} \tag{Q9.3.1}$$

to express Maxwell's equations in terms of $\underline{\underline{F}}$

(c) Use

$$*^{(4)}\left(\underline{e}^t \wedge \underline{E}\right) = - *^{(3)} \underline{E} \tag{Q9.3.2}$$

and

$$*^{(4)}\underline{\underline{B}} = \underline{e}^t \wedge *^{(3)}\underline{\underline{B}} \tag{Q9.3.3}$$

to show that

$$\underline{\underline{N}} = -\underline{e}^t \wedge \underline{M} + \underline{\underline{P}} \tag{Q9.3.4}$$

(d) Give the topological and physical meanings of $\underline{\nabla} \wedge \underline{\underline{N}}$ from both the four dimensional and three plus one dimensional points of view.

A9.3. (a) Maxwell's equations are

$$\begin{aligned} \underline{\nabla} \wedge \underline{\underline{B}} &= 0 \quad , \quad & \underline{\nabla} \wedge \underline{E} + \dot{\underline{\underline{B}}} &= 0 \\ \underline{\nabla} \wedge \underline{\underline{D}} &= \underline{\underline{\underline{\rho_{\mathrm{f}}}}} \quad , \quad & \underline{\nabla} \wedge \underline{H} - \dot{\underline{\underline{D}}} &= \underline{\underline{j_{\mathrm{f}}}} \end{aligned} \tag{A9.3.1}$$

Eqs. (2.1.55) and (2.1.56) are

$$\underline{\underline{D}} = *\underline{E} + \underline{\underline{P}} \tag{A9.3.2}$$

$$\underline{H} = *\underline{\underline{B}} - \underline{M} \tag{A9.3.3}$$

and Eq. (2.1.57) for an n-form $\boldsymbol{\omega}$ is

$$(-1)^{n-1}\boldsymbol{\nabla} \cdot \boldsymbol{\omega} = *^{-1}\boldsymbol{\nabla} \wedge *\boldsymbol{\omega} \tag{A9.3.4}$$

Substituting Eqs. (A9.3.2) and (A9.3.3) into Eq. (A9.3.1) and using Eq. (A9.3.4) gives

$$\begin{aligned} \underline{\nabla} \wedge \underline{\underline{B}} &= 0 \quad , \quad & \underline{\nabla} \wedge \underline{E} + \dot{\underline{\underline{B}}} &= 0 \\ \underline{\nabla} \cdot \underline{E} &= *^{-1}\underline{\underline{\underline{\rho_{\mathrm{t}}}}} \quad , \quad & -\underline{\nabla} \cdot \underline{\underline{B}} - \dot{\underline{E}} &= *^{-1}\underline{\underline{j_{\mathrm{t}}}} \end{aligned} \tag{A9.3.5}$$

where

$$\underline{\underline{\underline{\rho_{\mathrm{t}}}}} = \underline{\underline{\underline{\rho_{\mathrm{f}}}}} - \underline{\nabla} \wedge \underline{\underline{P}} \tag{A9.3.6}$$

$$\underline{\underline{j_{\mathrm{t}}}} = \underline{\underline{j_{\mathrm{f}}}} + \dot{\underline{\underline{P}}} + \underline{\nabla} \wedge \underline{M} \tag{A9.3.7}$$

(b) Maxwell's equations are

$$\underline{\nabla} \wedge \underline{\underline{F}} = 0 \quad \text{(A9.3.8)}$$
$$\underline{\nabla} \wedge \underline{\underline{G}} = \underline{\underline{\underline{J_\mathrm{f}}}} \quad \text{(A9.3.9)}$$

Substituting Eq. (Q9.3.1) and using Eq. (A9.3.4) gives

$$\underline{\nabla} \wedge \underline{\underline{F}} = 0 \quad \text{(A9.3.10)}$$
$$-\underline{\nabla} \cdot \underline{\underline{F}} = *^{-1} \underline{\underline{\underline{J_\mathrm{t}}}} \quad \text{(A9.3.11)}$$

where

$$\underline{\underline{\underline{J_\mathrm{t}}}} = \underline{\underline{\underline{J_\mathrm{f}}}} + \underline{\nabla} \wedge \underline{\underline{N}} \quad \text{(A9.3.12)}$$

(c) Using Eqs. (1.3.41), (2.1.55) and (2.1.56), (Q9.3.2) and (Q9.3.3), and (1.3.34)

$$\begin{aligned}
\underline{\underline{G}} &= -\underline{e}^t \wedge \underline{H} - \underline{\underline{D}} && \text{(A9.3.13)}\\
&= -\underline{e}^t \wedge *^{(3)}\underline{\underline{B}} + \underline{e}^t \wedge \underline{M} - *^{(3)}\underline{E} - \underline{\underline{P}} && \text{(A9.3.14)}\\
&= *^{(4)}\left(\underline{e}^t \wedge \underline{E}\right) - *^{(4)}\underline{\underline{B}} + \underline{e}^t \wedge \underline{M} - \underline{\underline{P}} && \text{(A9.3.15)}\\
&= *^{(4)}\underline{\underline{F}} + \underline{e}^t \wedge \underline{M} - \underline{\underline{P}} && \text{(A9.3.16)}
\end{aligned}$$

and comparing with Eq. (Q9.3.1) gives Eq. (Q9.3.4).

(d) **4D topological** $\underline{\underline{N}}$ is a two-form in four dimensions, and so is an oriented surface density[1]. $\underline{\nabla} \wedge \underline{\underline{N}}$ is the boundaries of the $\underline{\underline{N}}$ surfaces, and so is a flux density whose lines have no ends, i.e. a conserved flux density.

4D physical $\underline{\nabla} \wedge \underline{\underline{N}}$ appears as a source for $\underline{\underline{F}}$ in Eq. (A9.3.11) and can be interpreted as the bound current density

$$\underline{\underline{\underline{J_\mathrm{b}}}} = \underline{\nabla} \wedge \underline{\underline{N}} \quad \text{(A9.3.17)}$$

with the $\underline{\underline{N}}$ surfaces joining the worldlines of bound charge pairs or filling in bound charge worldloops. Note that

$$\underline{\nabla} \wedge \underline{\underline{\underline{J_\mathrm{b}}}} = 0 \quad \text{(A9.3.18)}$$

so that the free current $\underline{\underline{\underline{J_\mathrm{f}}}}$ and the bound current $\underline{\underline{\underline{J_\mathrm{b}}}}$ are independently conserved.

[1] A density of surfaces, not a density on a surface.

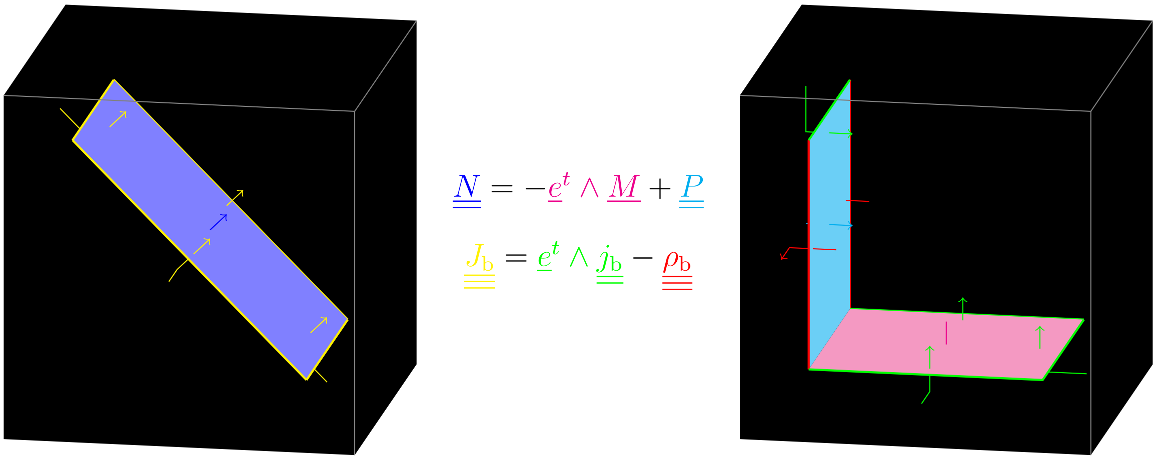

Figure A9.3.1: Space and time decomposition of $\underline{\nabla} \wedge \underline{\underline{N}} = \underline{\underline{\underline{J}}}_{\mathrm{b}}$ and $\underline{\nabla} \wedge \underline{\underline{\underline{J}}}_{\mathrm{b}} = 0$ to give $\underline{\nabla}^{(3)} \wedge \underline{\underline{P}} = -\underline{\underline{\underline{\rho}}}_{\mathrm{b}}$ and $-\underline{\nabla}^{(3)} \wedge (\underline{e}^t \wedge \underline{M}) + \underline{e}^t \wedge \dot{\underline{\underline{P}}} = \underline{e}^t \wedge \underline{\underline{j}}_{\mathrm{b}}$ and $-\underline{\nabla}^{(3)} \wedge \left(\underline{e}^t \wedge \underline{\underline{j}}_{\mathrm{b}}\right) + \underline{e}^t \wedge \dot{\underline{\underline{\underline{\rho}}}}_{\mathrm{b}} = 0$. One external space dimension has been suppressed.

(3+1)D topological Figure A9.3.1 illustrates

$$\begin{aligned} \underline{\underline{N}} &= -\underline{e}^t \wedge \underline{M} + \underline{\underline{P}} && \text{(A9.3.19)} \\ \underline{\underline{\underline{J}}}_{\mathrm{b}} &= \underline{e}^t \wedge \underline{\underline{j}}_{\mathrm{b}} - \underline{\underline{\underline{\rho}}}_{\mathrm{b}} && \text{(A9.3.20)} \end{aligned}$$

and

$$\underline{\nabla} \wedge \underline{\underline{N}} = \underline{\underline{\underline{J}}}_{\mathrm{b}}$$
$$\rightarrow \left\{ \begin{array}{rcl} \underline{\nabla}^{(3)} \wedge \underline{\underline{P}} = -\underline{\underline{\underline{\rho}}}_{\mathrm{b}} & \rightarrow & \underline{\nabla} \wedge \underline{\underline{P}} = -\underline{\underline{\underline{\rho}}}_{\mathrm{b}} \\ \underline{\nabla}^{(3)} \wedge \left(-\underline{e}^t \wedge \underline{M}\right) + \underline{e}^t \wedge \dot{\underline{\underline{P}}} = \underline{e}^t \wedge \underline{\underline{j}}_{\mathrm{b}} & \rightarrow & \underline{\nabla} \wedge \underline{M} + \dot{\underline{\underline{P}}} = \underline{\underline{j}}_{\mathrm{b}} \end{array} \right. \tag{A9.3.21}$$

and

$$\underline{\nabla} \wedge \underline{\underline{\underline{J}}}_{\mathrm{b}} = 0 \;\rightarrow\; \underline{\nabla}^{(3)} \wedge \left(-\underline{e}^t \wedge \underline{\underline{j}}_{\mathrm{b}}\right) + \underline{e}^t \wedge \dot{\underline{\underline{\underline{\rho}}}}_{\mathrm{b}} = 0 \;\rightarrow\; \underline{\nabla} \wedge \underline{\underline{j}}_{\mathrm{b}} + \dot{\underline{\underline{\underline{\rho}}}}_{\mathrm{b}} = 0 \tag{A9.3.22}$$

(3+1)D physical $\underline{\underline{P}}$ is a flux density corresponding to the electric dipole density

$$\underline{\underline{P}} = \underline{\underline{\underline{n}}}_{\mathrm{e}} \cdot \vec{p} \tag{A9.3.23}$$

where $\underline{\underline{\underline{n}}}_{\mathrm{e}}$ is the electric dipole number density and $\vec{p}$ is the electric dipole moment. The flux lines of $\underline{\underline{P}}$ can be interpreted as the oriented lines joining the bound charge pairs. If one cuts a flux line, a pair of opposite charges appears at the new ends but the electric dipole density does not change. Thus one can also connect up the flux segments, eliminating opposite charge pairs, to form continuous flux lines. $\underline{\nabla} \wedge \underline{\underline{P}}$ is the density of flux line end points, and so corresponds to the bound charge density

$$\underline{\underline{\underline{\rho_{\mathrm{b}}}}} = -\underline{\nabla} \wedge \underline{\underline{P}} \tag{A9.3.24}$$

$\underline{\underline{\dot{P}}}$ is the rate of increase of flux lines, which requires flux line end points to move past, and so $\underline{\underline{\dot{P}}}$ corresponds to a bound current density, see Eq. (A9.3.26). $\underline{M}$ is an oriented surface density corresponding to the magnetic dipole or current loop density

$$\underline{M} = \underline{\underline{\underline{n_{\mathrm{m}}}}} \cdot \vec{\vec{m}} \tag{A9.3.25}$$

where $\underline{\underline{\underline{n_{\mathrm{m}}}}}$ is the magnetic dipole number density and $\vec{\vec{m}}$ is the magnetic dipole moment. The surfaces of $\underline{M}$ can be interpreted as the oriented surfaces filling in the bound current loops. Note that if one cuts a surface, a pair of opposite currents appears at the new edges but the magnetic dipole density does not change. Thus one can also connect up the surface elements, eliminating opposite currents, to form continuous surfaces. $\underline{\nabla} \wedge \underline{M}$ is the density of surface edges, and so also contributes to the bound current density

$$\underline{\underline{j_{\mathrm{b}}}} = \underline{\underline{\dot{P}}} + \underline{\nabla} \wedge \underline{M} \tag{A9.3.26}$$