Chapter 3

Cosmology

3.1 Expanding universe

The observable universe is observed to be spatially isotropic, homogeneous and flat. Decomposing the metric as in Eq. (2.2.9), spatial isotropy implies $B_{\mathbf{a}} = 0$, spatial homogeneity implies $A = A(t)$ and $h_{\mathbf{ab}} = h_{\mathbf{ab}}(t)$ and hence we can redefine the time coordinate such that $A = 1$, and spatial isotropy, homogeneity and flatness imply $h_{\mathbf{ab}} \propto \delta_{\mathbf{ab}}$ where $\delta_{\mathbf{ab}}$ is the metric on three dimensional Euclidean space. Thus the metric for a spatially isotropic, homogeneous and flat universe takes the form

$$g_{\mathbf{ab}} = e^t_{\mathbf{a}} e^t_{\mathbf{b}} - a^2(t)\, \delta_{\mathbf{ab}} \tag{3.1.1}$$

or equivalently

$$d\tau^2 = g_{\mathbf{ab}} dx^{\mathbf{a}} dx^{\mathbf{b}} = dt^2 - a^2(t)\, \delta_{\mathbf{ab}} dx^{\mathbf{a}} dx^{\mathbf{b}} \tag{3.1.2}$$

where $a(t)$ is the **scale factor** and $\delta_{\mathbf{ab}}$ is the **comoving** metric. Similarly, the stress tensor takes the form

$$T_{\mathbf{ab}} = \rho(t)\, e^t_{\mathbf{a}} e^t_{\mathbf{b}} + p(t) \left(e^t_{\mathbf{a}} e^t_{\mathbf{b}} - g_{\mathbf{ab}} \right) \tag{3.1.3}$$

Substituting Eqs. (3.1.1) and (3.1.3) into the Einstein equation, Eq. (2.4.9), gives the **Friedmann equations**

$$3\frac{\dot{a}^2}{a^2} = \rho \tag{3.1.4}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p) \tag{3.1.5}$$

Restoring c and $8\pi G$, Eq. (3.1.5) can be rewritten in Newtonian style as

$$\ddot{a} = -\frac{G(\rho + 3p)\left(\frac{4}{3}\pi a^3\right)}{c^2 a^2} \tag{3.1.6}$$

with the relativistic source $(\rho + 3p)/c^2$ reducing to the mass density in the Newtonian limit.

Eq. (3.1.4) relates the curvature of spacetime, in the form of the expansion rate or **Hubble parameter**

$$H \equiv \frac{\dot{a}}{a} \tag{3.1.7}$$

to the energy density. Reexpressing Eqs. (3.1.4) and (3.1.5) in terms of the Hubble parameter gives

$$3H^2 = \rho \tag{3.1.8}$$

$$\dot{H} = -\frac{1}{2}(\rho + p) \tag{3.1.9}$$

and taking the derivative gives

$$\dot{\rho} = -3H(\rho + p) \tag{3.1.10}$$

which can also be derived directly from $\nabla_{\mathbf{b}} T_{\mathbf{a}}^{\mathbf{b}} = 0$.

From Eq. (3.1.2), physical spatial distances are given by

$$s = ax \tag{3.1.11}$$

where x is the comoving coordinate distance. Therefore, for objects moving with the expansion of the universe, i.e. with constant x, we have

$$\frac{ds}{dt} = \dot{a}x = \frac{\dot{a}}{a}s = Hs \tag{3.1.12}$$

which is **Hubble's law**: comoving objects are receding from us at a speed proportional to their distance from us. Further, at distances $s > 1/H$ we have $ds/dt > 1$ and so comoving objects are moving away from us at greater than the speed of light and so are out of causal contact and are said to be beyond the **horizon**. The **observable universe** corresponds to $s < 1/H$.