2.6 Black holes

2.6.1 Schwarzschild metric

The spacetime outside a star or planet is spherically symmetric and static[1]. Decomposing the metric as in Eq. (2.2.9), static implies $B_{\mathbf{a}} = 0$, stationary and spherically symmetric implies $A = A(r)$ and $h_{\mathbf{ab}} = h_{\mathbf{ab}}(r)$, and choosing the radial coordinate to be the areal radius gives $h_{\mathbf{ab}}\, dx^{\mathbf{a}}\, dx^{\mathbf{b}} = C(r)\, dr^2 + r^2\, d\Omega^2$. Thus the metric for a static spherically symmetric spacetime takes the form

$$d\tau^2 = A(r)\, dt^2 - C(r)\, dr^2 - r^2\, d\Omega^2 \tag{2.6.1}$$

Substituting into the vacuum Einstein equation and solving gives the **Schwarzschild metric**

$$d\tau^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2\, d\Omega^2 \tag{2.6.2}$$

Global structure

The metric Eq. (2.6.2) seems to be singular at $r = 2GM$, but introducing Kruskal coordinates

$$uv = \left(\frac{r}{2GM} - 1\right) \exp\left(\frac{r}{2GM}\right) \tag{2.6.3}$$

$$\frac{u}{v} = \exp\left(\frac{t}{2GM}\right) \tag{2.6.4}$$

which are constant on in-going and out-going radial light rays respectively, the metric becomes

$$d\tau^2 = \frac{32G^3M^3}{r} \exp\left(-\frac{r}{2GM}\right) du\, dv - r^2\, d\Omega^2 \tag{2.6.5}$$

and we see that $r = 2GM$ is just a coordinate singularity, the only physical singularity being at $r = 0$. The global structure of Schwarzschild spacetime, including the **event horizon** ar $r = 2GM$, can be illustrated in a Penrose diagram. The coordinates

$$u' = \tan^{-1} u \tag{2.6.6}$$

$$v' = \tan^{-1} v \tag{2.6.7}$$

bring infinity in to finite values, and a conformal rescaling of the metric

$$g_{\mathbf{ab}} \to \Phi(x)\, g_{\mathbf{ab}} \tag{2.6.8}$$

generates Figure 2.6.1.

[1] Birkhoff's theorem states that a spherically symmetric vacuum spacetime is necessarily static. Note that (static $\equiv$ stationary + time reversal symmetric) and (stationary $\equiv$ time independent).

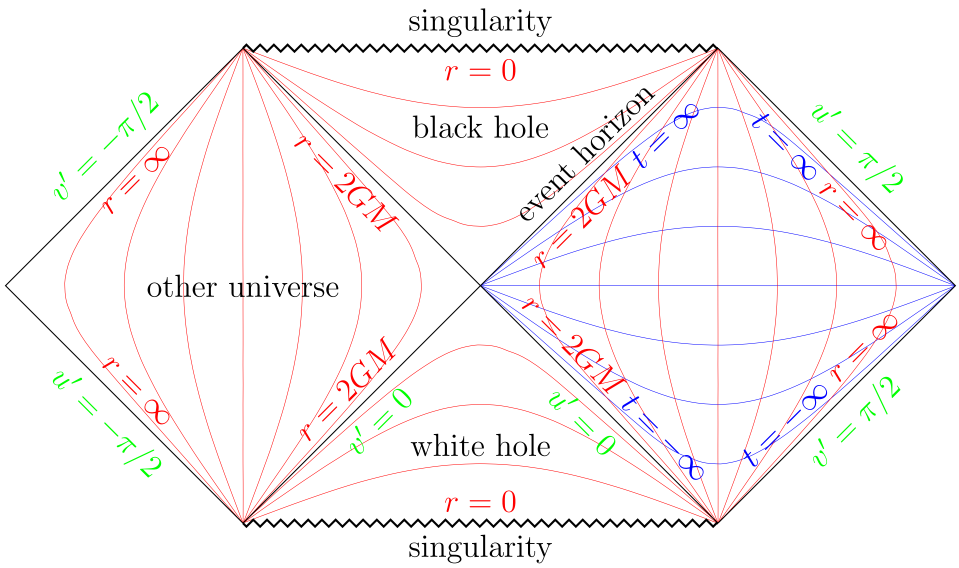

Figure 2.6.1: Penrose diagram for Schwarzschild spacetime. The white hole and other universe regions are unphysical consequences of the assumed static, or equivalently vacuum, nature of the spacetime. The singularity indicates that general relativity is breaking down and needs to be replaced by a more complete theory. Time-like infinity, $(u', v') = (\pi/2, 0)$, is modified by black hole evaporation.

Orbits

Fixing $\theta = \pi/2$ and using the results of Section 1.2.3, time translational and rotational symmetries give the conserved quantities

$$E = e_t^{\mathbf{a}} p_{\mathbf{a}} = m g_{t\mathbf{b}} \frac{dx^{\mathbf{b}}}{d\tau} = m\left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau} \qquad (2.6.9)$$

and

$$L = -e_\phi^{\mathbf{a}} p_{\mathbf{a}} = -m g_{\phi\mathbf{b}} \frac{dx^{\mathbf{b}}}{d\tau} = m r^2 \frac{d\phi}{d\tau} \qquad (2.6.10)$$

while the radial motion can be determined using

$$m^2 = g^{\mathbf{ab}} p_{\mathbf{a}} p_{\mathbf{b}} = g^{tt} p_t^2 + g^{rr} p_r^2 + g^{\phi\phi} p_\phi^2 \qquad (2.6.11)$$

$$= \left(1 - \frac{2GM}{r}\right)^{-1} E^2 - m^2 \left(1 - \frac{2GM}{r}\right)^{-1} \left(\frac{dr}{d\tau}\right)^2 - \frac{L^2}{r^2} \qquad (2.6.12)$$

therefore

$$\frac{1}{2} m \left(\frac{dr}{d\tau}\right)^2 + V(r) = \frac{E^2 - m^2}{2m} \qquad (2.6.13)$$

where the effective potential

$$V(r) = -\frac{GMm}{r} + \frac{L^2}{2mr^2}\left(1 - \frac{2GM}{r}\right) \qquad (2.6.14)$$

has extrema at

$$r_\pm = \frac{L^2 \pm \sqrt{L^4 - 12G^2M^2m^2L^2}}{2GMm^2} \tag{2.6.15}$$

corresponding to stable and unstable circular orbits.

Photon orbits can be obtained using the worldline parameter $d\lambda = d\tau/m$ and taking $m \to 0$.

2.6.2 Kerr metric

The metric for a stationary axially symmetric, i.e. rotating, black hole is

$$\begin{aligned} d\tau^2 &= \left(1 - \frac{2GMr}{\rho^2}\right) dt^2 + 2\left(\frac{2GMra\sin^2\theta}{\rho^2}\right) dt\, d\phi - \left(\frac{\rho^2}{r^2 - 2GMr + a^2}\right) dr^2 \\ &\quad - \rho^2\, d\theta^2 - \left(r^2 + a^2 + \frac{2GMra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta\, d\phi^2 \end{aligned} \tag{2.6.16}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta \tag{2.6.17}$$

and $a = J/M$ with M and J the mass and angular momentum of the black hole. There are two important surfaces, the **ergosphere** where g_{tt} passes through zero

$$r_{\mathrm{E}} = GM + \sqrt{G^2M^2 - a^2\cos^2\theta} \tag{2.6.18}$$

and the **horizon** where g_{rr} diverges

$$r_{\mathrm{H}} = GM + \sqrt{G^2M^2 - a^2} \tag{2.6.19}$$

Between these surfaces, a timelike curve must have $d\phi/dt > 0$, i.e. a particle must rotate with the black hole.

The limiting case of $a = GM$ is an extreme black hole, and for $a > GM$ the horizon disappears leaving a naked singularity.

Equatorial orbits

Orbits with $\theta = \pi/2$ can be determined in the same way as for the Schwarzschild metric, the effective potential Eq. (2.6.14) becoming

$$V(r) = -\frac{GMm}{r} + \frac{L^2 - a^2\left(E^2 - m^2\right)}{2mr^2} - \frac{GM\left(L - aE\right)^2}{mr^3} \tag{2.6.20}$$