2.3 Particles in spacetime

A **particle** is something that exists as a **worldline** in spacetime.

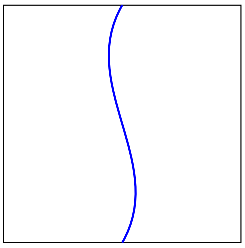

Figure 2.3.1: A particle in spacetime.

2.3.1 Kinematics

We can parameterize a particle's worldline by its length to define a physical velocity

$$u^{\mathbf{a}} = \frac{dx^{\mathbf{a}}}{d\tau} \tag{2.3.1}$$

and acceleration

$$a^{\mathbf{a}} = \frac{d^2x^{\mathbf{a}}}{d\tau^2} \tag{2.3.2}$$

The magnitude of the velocity

$$g_{\mathbf{ab}}u^{\mathbf{a}}u^{\mathbf{b}} = \frac{g_{\mathbf{ab}}dx^{\mathbf{a}}dx^{\mathbf{b}}}{d\tau^2} = 1 \tag{2.3.3}$$

and hence the acceleration is always perpendicular to the velocity

$$g_{\mathbf{ab}}u^{\mathbf{a}}a^{\mathbf{b}} = 0 \tag{2.3.4}$$

2.3.2 Action

A worldline C in a spacetime M has action

$$-S[C] = \int_C (m\underline{\sigma} + q\underline{A}) \tag{2.3.5}$$

where the worldline volume form $\underline{\sigma}$ measures the length along the curve

$$d\tau = \underline{\sigma} \cdot \vec{dx} \tag{2.3.6}$$

and $\underline{A}$ is a covector field in the spacetime. Note that the physics given by $\delta S = 0$ is invariant under

$$\underline{A} \to \underline{A} + \underline{\nabla}\lambda \tag{2.3.7}$$

since

$$\int_C \underline{\nabla}\lambda = \int_{\partial C} \lambda \tag{2.3.8}$$

is a boundary term. In Lagrangian form

$$\begin{aligned} -S &= \int_C \left(m\sigma_{\mathbf{a}} + qA_{\mathbf{a}}\right) dx^{\mathbf{a}} & (2.3.9) \\ &= \int_C \left(m\sqrt{g_{\mathbf{ab}}\dot{x}^{\mathbf{a}}\dot{x}^{\mathbf{b}}} + qA_{\mathbf{a}}\dot{x}^{\mathbf{a}}\right) dt & (2.3.10) \end{aligned}$$

The Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^{\mathbf{a}}}\right) = \frac{\partial L}{\partial x^{\mathbf{a}}} \tag{2.3.11}$$

gives

$$\frac{d}{dt}\left(p_{\mathbf{a}} + qA_{\mathbf{a}}\right) = q\left(\nabla_{\mathbf{a}} A_{\mathbf{b}}\right)\frac{dx^{\mathbf{b}}}{dt} \tag{2.3.12}$$

where the particle's momentum [1]

$$p_{\mathbf{a}} = \frac{m g_{\mathbf{ab}}}{\sqrt{g_{\mathbf{cd}}\dot{x}^{\mathbf{c}}\dot{x}^{\mathbf{d}}}}\frac{dx^{\mathbf{b}}}{dt} = m g_{\mathbf{ab}}\frac{dx^{\mathbf{b}}}{d\tau} \tag{2.3.13}$$

Therefore the force on the particle is

$$f_{\mathbf{a}} = \frac{dp_{\mathbf{a}}}{d\tau} = m g_{\mathbf{ab}}\frac{d^2 x^{\mathbf{b}}}{d\tau^2} = qF_{\mathbf{ab}}\frac{dx^{\mathbf{b}}}{d\tau} \tag{2.3.14}$$

where the electromagnetic field

$$F_{\mathbf{ab}} = \nabla_{\mathbf{a}} A_{\mathbf{b}} - \nabla_{\mathbf{b}} A_{\mathbf{a}} \tag{2.3.15}$$

2.3.3 Newtonian decomposition

Following Eq. (2.2.7), the spacetime velocity decomposes as

$$u^{\mathbf{a}} = \frac{dx^{\mathbf{a}}}{d\tau} = \frac{dt}{d\tau}\left(e_t^{\mathbf{a}} + v^{\mathbf{a}}\right) \tag{2.3.16}$$

where the spatial velocity

$$v^{\mathbf{a}} = \frac{dx_3^{\mathbf{a}}}{dt} \tag{2.3.17}$$

the spacetime force decomposes as

$$f_{\mathbf{a}} = \frac{dt}{d\tau}\left(P e_{\mathbf{a}}^t - F_{\mathbf{a}}\right) \tag{2.3.18}$$

[1] Note that $p_{\mathbf{a}} = m\sigma_{\mathbf{a}}$.

where the spatial force satisfies

$$e_t^{\mathbf{a}} F_{\mathbf{a}} = 0 \tag{2.3.19}$$

and the electromagnetic field decomposes as

$$F_{\mathbf{ab}} = e_{\mathbf{a}}^t E_{\mathbf{b}} - E_{\mathbf{a}} e_{\mathbf{b}}^t - B_{\mathbf{ab}} \tag{2.3.20}$$

where the electric field $E_{\mathbf{a}}$ and magnetic flux density $B_{\mathbf{ab}}$ satisfy

$$e_t^{\mathbf{a}} E_{\mathbf{a}} = 0 \tag{2.3.21}$$

$$e_t^{\mathbf{a}} B_{\mathbf{ab}} = 0 \tag{2.3.22}$$

Substituting Eqs. (2.3.16), (2.3.18) and (2.3.20) into Eq. (2.3.14) gives

$$P e_{\mathbf{a}}^t - F_{\mathbf{a}} = q \left(e_{\mathbf{a}}^t E_{\mathbf{b}} - E_{\mathbf{a}} e_{\mathbf{b}}^t - B_{\mathbf{ab}} \right) \left(e_t^{\mathbf{b}} + v^{\mathbf{b}} \right) \tag{2.3.23}$$

$$= q \left(e_{\mathbf{a}}^t E_{\mathbf{b}} v^{\mathbf{b}} - E_{\mathbf{a}} - B_{\mathbf{ab}} v^{\mathbf{b}} \right) \tag{2.3.24}$$

Therefore Eq. (2.3.14) decomposes into the electromagnetic power equation

$$P = q E_{\mathbf{a}} v^{\mathbf{a}} \tag{2.3.25}$$

and the Lorentz force law

$$F_{\mathbf{a}} = q \left(E_{\mathbf{a}} + B_{\mathbf{ab}} v^{\mathbf{b}} \right) \tag{2.3.26}$$