2.2 Spacetime

The central idea of relativity is that space and time are unified into **spacetime**.

2.2.1 Structure of spacetime

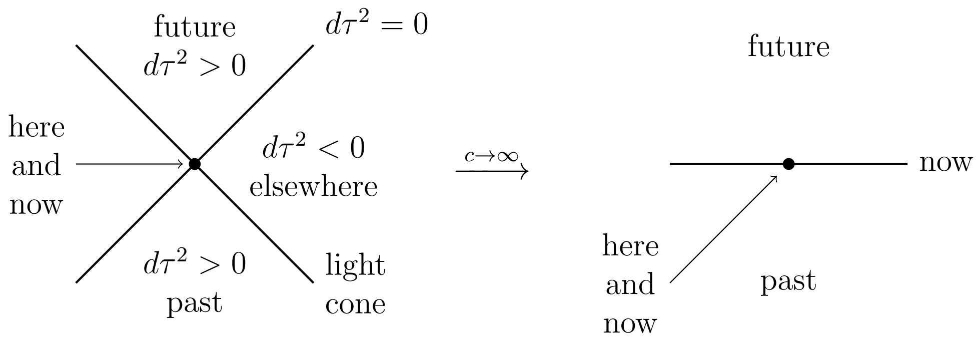

Figure 2.2.1: Relativistic spacetime and its Newtonian limit.

In Minkowski coordinates, an infinitesimal displacement squared can be expressed in terms of the **proper time** τ

$$d\tau^2 = dt^2 - \frac{1}{c^2}\left(dx^2 + dy^2 + dz^2\right) \tag{2.2.1}$$

or equivalently in terms of the **proper distance** s

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2 \tag{2.2.2}$$

where the minus sign allows us to distinguish time-like and space-like directions, see Figure 2.2.1. Note that only $d\tau^2$ or ds^2 is physical while dt^2 and $dx^2 + dy^2 + dz^2$ are coordinate dependent. In the Newtonian limit these reduce to

$$d\tau^2\big|_{c\to\infty} = dt^2 \tag{2.2.3}$$

$$ds^2\big|_{dt=0} = \left(dx^2 + dy^2 + dz^2\right)_{dt=0} \tag{2.2.4}$$

2.2.2 Spacetime decomposition

To view things from our usual Newtonian space and time perspective, we introduce a Newtonian time coordinate t which defines a set of spatial hypersurfaces and a corresponding covector field

$$e^t_{\mathbf{a}} = \nabla_{\mathbf{a}} t \tag{2.2.5}$$

We also introduce a spatial rest-frame which defines a fibration of the hypersurfaces by one dimensional time lines and a corresponding vector field $e_t^{\mathbf{a}}$ satisfying

$$e_t^{\mathbf{a}} e^t_{\mathbf{a}} = 1 \tag{2.2.6}$$

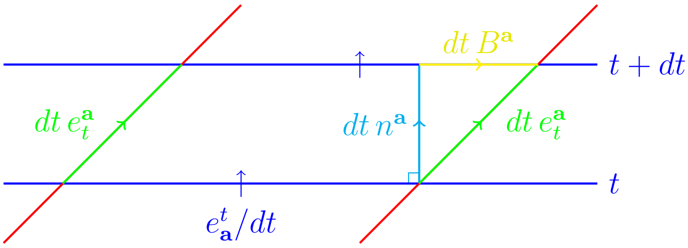

Figure 2.2.2: Fibration of spatial hypersurfaces and corresponding time vector $e_t^{\mathbf{a}}$ and covector $e^t_{\mathbf{a}}$. The time vector $e_t^{\mathbf{a}}$ decomposes into the normal $n^{\mathbf{a}}$ and shift $B^{\mathbf{a}}$ vectors.

see Figure 2.2.2.

Now we can decompose a spacetime displacement as

$$dx^{\mathbf{a}} = dt\, e_t^{\mathbf{a}} + dx_3^{\mathbf{a}} \tag{2.2.7}$$

with the spatial displacement satisfying

$$e^t_{\mathbf{a}} dx_3^{\mathbf{a}} = 0 \tag{2.2.8}$$

i.e. spatial displacements lie within the spatial hypersurfaces.

The metric is decomposed as

$$g_{\mathbf{ab}} = A e^t_{\mathbf{a}} e^t_{\mathbf{b}} - B_{\mathbf{a}} e^t_{\mathbf{b}} - e^t_{\mathbf{a}} B_{\mathbf{b}} - h_{\mathbf{ab}} \tag{2.2.9}$$

with the shift covector satisfying

$$e_t^{\mathbf{a}} B_{\mathbf{a}} = 0 \tag{2.2.10}$$

and the spatial metric satisfying

$$e_t^{\mathbf{a}} h_{\mathbf{ab}} = 0 \tag{2.2.11}$$

giving

$$d\tau^2 = g_{\mathbf{ab}}\, dx^{\mathbf{a}}\, dx^{\mathbf{b}} = A\, dt^2 - 2B_{\mathbf{a}}\, dx_3^{\mathbf{a}}\, dt - h_{\mathbf{ab}}\, dx_3^{\mathbf{a}}\, dx_3^{\mathbf{b}} \tag{2.2.12}$$

The inverse metric is decomposed as

$$g^{\mathbf{ab}} = \frac{\left(e_t^{\mathbf{a}} - B^{\mathbf{a}}\right)\left(e_t^{\mathbf{b}} - B^{\mathbf{b}}\right)}{A + B^{\mathbf{c}} B_{\mathbf{c}}} - h^{\mathbf{ab}} \tag{2.2.13}$$

where the inverse spatial metric $h^{\mathbf{ab}}$ is defined by

$$h^{\mathbf{ab}} h_{\mathbf{bc}} = \delta^{\mathbf{a}}_{\mathbf{c}} - e_t^{\mathbf{a}} e^t_{\mathbf{c}} \tag{2.2.14}$$

and

$$e^t_{\mathbf{a}} h^{\mathbf{ab}} = 0 \tag{2.2.15}$$

and the shift vector

$$B^{\mathbf{a}} \equiv h^{\mathbf{ab}} B_{\mathbf{b}} \tag{2.2.16}$$

so that

$$e^t_{\mathbf{a}} B^{\mathbf{a}} = 0 \tag{2.2.17}$$

The normal vector to the spatial hypersurfaces

$$n^{\mathbf{a}} = \frac{g^{\mathbf{ab}} e^t_{\mathbf{b}}}{g^{\mathbf{cd}} e^t_{\mathbf{c}} e^t_{\mathbf{d}}} \tag{2.2.18}$$

satisfies

$$n^{\mathbf{a}} e^t_{\mathbf{a}} = 1 \tag{2.2.19}$$

and

$$n^{\mathbf{a}} g_{\mathbf{ab}} dx_3^{\mathbf{b}} = 0 \tag{2.2.20}$$

for any spatial displacement $dx_3^{\mathbf{a}}$. The time vector has magnitude squared

$$g_{\mathbf{ab}} e_t^{\mathbf{a}} e_t^{\mathbf{b}} = A \tag{2.2.21}$$

and decomposes into the normal and shift vectors

$$e_t^{\mathbf{a}} = n^{\mathbf{a}} + B^{\mathbf{a}} \tag{2.2.22}$$

see Figure 2.2.2.