1.2 Spaces

1.2.1 Tensor fields

Finite displacements in Euclidean space can be represented by arrows and have a natural vector space structure, but finite displacements in more general curved spaces, such as on the surface of a sphere, do not. However, an infinitesimal neighborhood of a point in a smooth curved space[1] looks like an infinitesimal neighborhood of Euclidean space, and infinitesimal displacements $\vec{dx}$ retain the vector space structure of displacements in Euclidean space. An infinitesimal neighborhood of a point can be infinitely rescaled to generate a finite vector space, called the **tangent space**, at the point. Note that

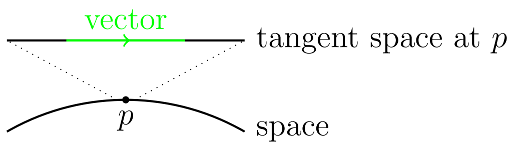

Figure 1.2.1: A vector in the tangent space of a point.

vectors do not stretch from one point to another, but instead live in the tangent space of a point, and vectors at different points live in different tangent spaces and so cannot be added.

For example, rescaling the infinitesimal displacement $\vec{dx}$ by dividing it by the infinitesimal scalar dt gives the velocity

$$\vec{v} = \frac{\vec{dx}}{dt} \tag{1.2.1}$$

which is a vector. Similarly, we can picture the covector $\underline{\nabla}\phi$ as the infinitesimal contours of ϕ in a neighborhood of a point, infinitely rescaled to generate a finite covector in the point's cotangent space. More generally, infinitely rescaling the neighborhood of a point generates the **tensor space** and its algebra at the point. The tensor space contains the tangent and cotangent spaces as vector subspaces.

A **tensor field** is something that takes tensor values at every point in a space. Tensor fields of the same type can be added, and multiplied by a scalar, in the usual way.

1.2.2 Tensor calculus

Integration

A covector field $\underline{\omega}$ naturally contracts with a curve C to give a scalar

$$\int_C \underline{\omega} = \text{scalar} \tag{1.2.2}$$

[1] In mathematical language, a smooth manifold.

with the same interpretation as the contraction of a covector with a vector. If we divide the curve C into infinitesimal line elements $\vec{dx}$, the integral of $\underline{\omega}$ over C can be written in the more familiar form

$$\int_C \underline{\omega} \cdot \vec{dx} \tag{1.2.3}$$

Covariant derivative

The **covariant derivative** $\nabla_{\mathbf{a}}$ is a derivative operator and hence is linear and obeys the Leibnitz rule. Its action on a scalar is given by Eq. (1.1.8)

$$dx^{\mathbf{a}} \nabla_{\mathbf{a}} \phi = d\phi \tag{1.2.4}$$

and it has the key property

$$\nabla_{\mathbf{a}} g_{\mathbf{bc}} = 0 \tag{1.2.5}$$

since the metric is used to measure changes. One more condition is needed to uniquely define the covariant derivative, the zero torsion condition

$$\left(\nabla_{\mathbf{a}} \nabla_{\mathbf{b}} - \nabla_{\mathbf{b}} \nabla_{\mathbf{a}}\right) \phi = 0 \tag{1.2.6}$$

which has the geometrical interpretation that parallel transported vectors, i.e. vectors transported such that their covariant derivative is zero, form closed parallelograms.

Curvature tensor

Covariant derivatives do not commute when acting on tensors

$$\left(\nabla_{\mathbf{a}} \nabla_{\mathbf{b}} - \nabla_{\mathbf{b}} \nabla_{\mathbf{a}}\right) w_{\mathbf{c}} = R_{\mathbf{abc}}{}^{\mathbf{d}} w_{\mathbf{d}} \tag{1.2.7}$$

since parallel transport of tensors in a curved space is path dependent. $R_{\mathbf{abcd}}$ is the **curvature tensor**, which is zero if and only if the space is flat.

The curvature tensor has the following symmetries

$$R_{\mathbf{bacd}} = -R_{\mathbf{abcd}} \tag{1.2.8}$$

$$R_{\mathbf{abdc}} = -R_{\mathbf{abcd}} \tag{1.2.9}$$

$$R_{[\mathbf{abc}]\mathbf{d}} = 0 \tag{1.2.10}$$

which imply

$$R_{\mathbf{cdab}} = R_{\mathbf{abcd}} \tag{1.2.11}$$

The curvature tensor can be contracted to give the **Ricci tensor**

$$R_{\mathbf{ac}} = R_{\mathbf{abc}}{}^{\mathbf{b}} \tag{1.2.12}$$

and **Ricci scalar**

$$R = R^{\mathbf{a}}_{\mathbf{a}} \tag{1.2.13}$$

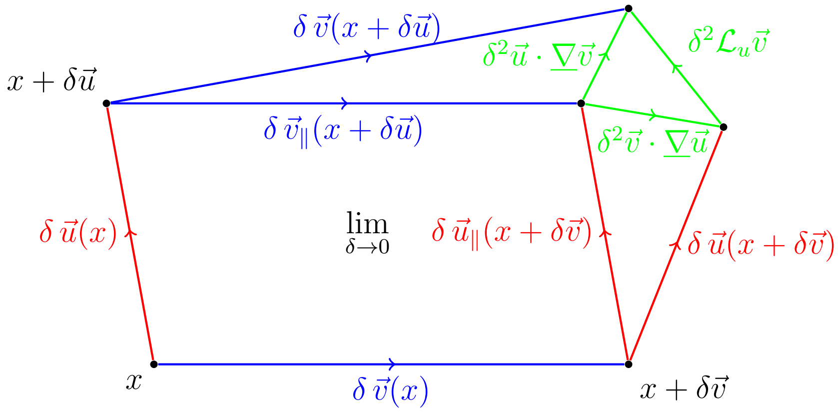

Figure 1.2.2: The Lie derivative and its relation to the covariant derivative. $\vec{v}_\parallel$ is $\vec{v}(x)$ parallel transported along $\vec{u}$, i.e. transported such that $\vec{u}\cdot\underline{\nabla}\vec{v}_\parallel = 0$, and $\vec{u}_\parallel$ is $\vec{u}(x)$ parallel transported along $\vec{v}$.

Lie derivative

The **Lie derivative**, with respect to a vector field $u^{\mathbf{a}}$, acting on a vector field $v^{\mathbf{a}}$, is

$$\mathcal{L}_u v^{\mathbf{a}} = u^{\mathbf{b}}\nabla_{\mathbf{b}}v^{\mathbf{a}} - v^{\mathbf{b}}\nabla_{\mathbf{b}}u^{\mathbf{a}} \tag{1.2.14}$$

It is the derivative relative to the flow generated by $u^{\mathbf{a}}$, see Figure 1.2.2. If $\mathcal{L}_u v^{\mathbf{a}} = 0$ then $u^{\mathbf{a}}$ and $v^{\mathbf{a}}$ are said to **commute**, since the vectors form closed quadrilaterals. Note that $\mathcal{L}_u$ depends on $u^{\mathbf{a}}$ and its derivative, but is independent of the metric.

1.2.3 Bases and coordinates

It is often convenient to choose a complete set of independent **basis** vectors $e_\alpha^{\mathbf{a}}$, where $\alpha = 1,\ldots,N$ labels the basis vectors and N is the dimension of the space, and express a general vector $v^{\mathbf{a}}$ as a linear combination of the basis vectors

$$v^{\mathbf{a}} = \sum_{\alpha=1}^{N} v^\alpha e_\alpha^{\mathbf{a}} \tag{1.2.15}$$

The scalars v^α are the **components** of the vector $v^{\mathbf{a}}$ and depend on the choice of basis $e_\alpha^{\mathbf{a}}$. We will use the summation convention for repeated component indices, so that the summation sign above is not explicitly written

$$v^{\mathbf{a}} = v^\alpha e_\alpha^{\mathbf{a}} \tag{1.2.16}$$

Note the difference between the two index notations. The abstract index **a** denotes the vector nature of $v^{\mathbf{a}}$, while the component index α labels the components v^α and basis vectors $e_\alpha^{\mathbf{a}}$, and is summed over in the above equation.

A vector basis $e_\alpha^{\mathbf{a}}$ naturally induces a covector basis $e_{\mathbf{a}}^\alpha$, or vice versa, via

$$e_{\mathbf{a}}^\alpha e_\beta^{\mathbf{a}} = \delta_\beta^\alpha \tag{1.2.17}$$

while completeness of the bases is expressed as

$$e^{\mathbf{a}}_{\alpha} e^{\alpha}_{\mathbf{b}} = \delta^{\mathbf{a}}_{\mathbf{b}} \tag{1.2.18}$$

A covector is expressed in components as

$$\omega_{\mathbf{a}} = \omega_{\alpha} e^{\alpha}_{\mathbf{a}} \tag{1.2.19}$$

and a covector contracted with a vector as

$$\omega_{\mathbf{a}} v^{\mathbf{a}} = \omega_{\alpha} v^{\alpha} \tag{1.2.20}$$

The inner product of basis vectors

$$\vec{e}_{\alpha} \cdot \vec{e}_{\beta} = g_{\mathbf{ab}} e^{\mathbf{a}}_{\alpha} e^{\mathbf{b}}_{\beta} = g_{\alpha\beta} \tag{1.2.21}$$

gives the metric components. An **orthonormal basis** has metric components $g_{\alpha\beta} = \delta_{\alpha\beta}$.

A coordinate system x^{α} induces a covector **coordinate basis** via

$$e^{\alpha}_{\mathbf{a}} = \nabla_{\mathbf{a}} x^{\alpha} \tag{1.2.22}$$

and the corresponding vector coordinate basis induced by Eq. (1.2.17) expands an infinitesimal displacement as

$$dx^{\mathbf{a}} = dx^{\alpha} e^{\mathbf{a}}_{\alpha} \tag{1.2.23}$$

where dx^{α} is the infinitesimal change in the coordinate x^{α}. Inverting Eq. (1.2.23) gives

$$e^{\mathbf{a}}_{\alpha} = \frac{\partial x^{\mathbf{a}}}{\partial x^{\alpha}} \tag{1.2.24}$$

Note that a coordinate basis covector $e^{\alpha}_{\mathbf{a}}$ is defined purely in terms of its coordinate x^{α}, with its plane tangent to the constant x^{α} surfaces and its magnitude given by the density of the surfaces, while a coordinate basis vector $e^{\mathbf{a}}_{\alpha}$ requires the full coordinate system, with its line tangent to the intersection of the constant x^{β}, $\beta \neq \alpha$, surfaces and its magnitude given by the separation of the x^{α} surfaces. See Figure 1.2.3.

The partial derivative with respect to a coordinate is the derivative in the direction in which the other coordinates are constant and is given by the corresponding basis vector

$$\frac{\partial}{\partial x^{\alpha}} = e^{\mathbf{a}}_{\alpha} \nabla_{\mathbf{a}} \tag{1.2.25}$$

In a coordinate system x^{α}, the length ds of an infinitesimal displacement $dx^{\mathbf{a}}$ can be expressed as

$$ds^2 = g_{\mathbf{ab}} dx^{\mathbf{a}} dx^{\mathbf{b}} = g_{\alpha\beta} dx^{\alpha} dx^{\beta} \tag{1.2.26}$$

For example, in Cartesian coordinates in two dimensional Euclidean space

$$ds^2 = dx^2 + dy^2 \tag{1.2.27}$$

and so the components of the metric are $g_{xx} = g_{yy} = 1$ and $g_{xy} = 0$. In polar coordinates

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{1.2.28}$$

and so $g_{rr} = 1$, $g_{\theta\theta} = r^2$ and $g_{r\theta} = 0$. **Cartesian bases** are the only orthonormal coordinate bases and they exist only in flat spaces.

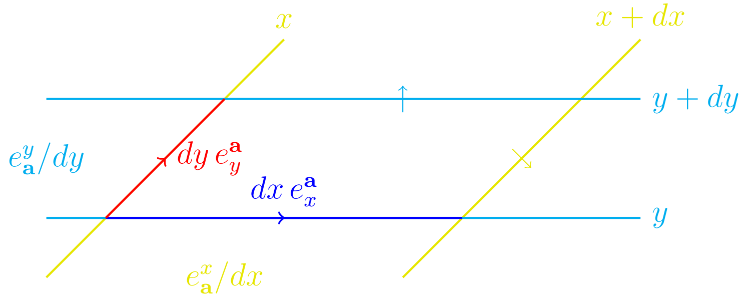

Figure 1.2.3: The coordinate basis covectors $e^x_{\mathbf{a}}$ and $e^y_{\mathbf{a}}$ are given by the x and y contours respectively. The coordinate basis vectors $e^{\mathbf{a}}_x$ and $e^{\mathbf{a}}_y$ lie along the y and x contours respectively, and span the x and y contours respectively.

Christoffel symbols

In a coordinate basis, the covariant derivative acting on a scalar has components

$$\nabla_{\mathbf{a}}\phi = \frac{\partial \phi}{\partial x^\alpha} e^\alpha_{\mathbf{a}} \tag{1.2.29}$$

Acting on a vector

$$\nabla_{\mathbf{a}} v^{\mathbf{c}} = \nabla_{\mathbf{a}}\left(v^\beta e^{\mathbf{c}}_\beta\right) = \left(\nabla_{\mathbf{a}} v^\beta\right) e^{\mathbf{c}}_\beta + v^\beta \nabla_{\mathbf{a}} e^{\mathbf{c}}_\beta \tag{1.2.30}$$

$\nabla_{\mathbf{a}} v^\beta$ is given by Eq. (1.2.29) and we define

$$\nabla_{\mathbf{a}} e^{\mathbf{c}}_\beta \equiv \Gamma^{\mathbf{c}}_{\mathbf{a}\beta} = \Gamma^\gamma_{\alpha\beta} e^\alpha_{\mathbf{a}} e^{\mathbf{c}}_\gamma \tag{1.2.31}$$

where the $\Gamma^\gamma_{\alpha\beta}$ are the **Christoffel symbols** [2]. Then

$$\nabla_{\mathbf{a}} v^{\mathbf{c}} = \left(\frac{\partial v^\gamma}{\partial x^\alpha} + \Gamma^\gamma_{\alpha\beta} v^\beta\right) e^\alpha_{\mathbf{a}} e^{\mathbf{c}}_\gamma \tag{1.2.32}$$

$$\nabla_{\mathbf{a}} \omega_{\mathbf{b}} = \left(\frac{\partial \omega_\beta}{\partial x^\alpha} - \Gamma^\gamma_{\alpha\beta} \omega_\gamma\right) e^\alpha_{\mathbf{a}} e^\beta_{\mathbf{b}} \tag{1.2.33}$$

and similarly for other tensors. Applying this to Eq. (1.2.5) we get an equation involving the first derivatives of the metric components and the Christoffel symbols which can be inverted using Eq. (1.2.6) to give

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta}\left(\frac{\partial g_{\delta\alpha}}{\partial x^\beta} + \frac{\partial g_{\delta\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\delta}\right) \tag{1.2.34}$$

[2] Note that $\Gamma^{\mathbf{c}}_{\mathbf{ab}} = \Gamma^\gamma_{\alpha\beta} e^\alpha_{\mathbf{a}} e^\beta_{\mathbf{b}} e^{\mathbf{c}}_\gamma$ is a tensor derived from a particular basis and so is a basis dependent tensor.

Geodesics

The motion of a particle in a space is described by a curve $x(t)$. The particle's velocity

$$v^{\mathbf{a}} = \frac{dx^{\mathbf{a}}}{dt} = \frac{dx^\alpha}{dt} e^{\mathbf{a}}_\alpha \tag{1.2.35}$$

is the tangent vector to the curve. The particle's **acceleration** is

$$a^{\mathbf{a}} = \frac{dv^{\mathbf{a}}}{dt} = \left(\frac{dv^\alpha}{dt} + \Gamma^\alpha_{\beta\gamma} v^\beta v^\gamma \right) e^{\mathbf{a}}_\alpha \tag{1.2.36}$$

where we have used

$$\frac{de^{\mathbf{a}}_\alpha}{dt} = v^{\mathbf{b}} \nabla_{\mathbf{b}} e^{\mathbf{a}}_\alpha = v^\beta \Gamma^\gamma_{\beta\alpha} e^{\mathbf{a}}_\gamma \tag{1.2.37}$$

Note that the velocity is the ratio of the infinitesimal vector $dx^{\mathbf{a}}$ and the infinitesimal scalar dt, while the acceleration is the derivative of the vector $v^{\mathbf{a}}$. If the acceleration is proportional to the velocity then the particle follows a **geodesic**, the generalization of a straight line to curved spaces. Further, if the parameterization of the curve is chosen such that the acceleration is zero then the parameter is **affine** and is proportional to the length along the curve. Thus an affinely parameterized geodesic $x(s)$ obeys the **geodesic equation**

$$\frac{d^2 x^{\mathbf{a}}}{ds^2} = \left(\frac{d^2 x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{ds} \frac{dx^\gamma}{ds} \right) e^{\mathbf{a}}_\alpha = 0 \tag{1.2.38}$$

Symmetry

A continuous symmetry is described by the flow generated by a vector field. If a vector field $\xi^{\mathbf{a}}$ satisfies **Killing's equation**

$$\begin{aligned} \mathcal{L}_\xi g_{\mathbf{ab}} &= \xi^{\mathbf{c}} \nabla_{\mathbf{c}} g_{\mathbf{ab}} + (\nabla_{\mathbf{a}} \xi^{\mathbf{c}}) g_{\mathbf{cb}} + (\nabla_{\mathbf{b}} \xi^{\mathbf{c}}) g_{\mathbf{ac}} && (1.2.39) \\ &= \nabla_{\mathbf{a}} \xi_{\mathbf{b}} + \nabla_{\mathbf{b}} \xi_{\mathbf{a}} = 0 && (1.2.40) \end{aligned}$$

then $\xi^{\mathbf{a}}$ is a **Killing vector field** and generates an isometry of the space. Now, in a coordinate basis,

$$\mathcal{L}_{e_\alpha} e^\beta_{\mathbf{b}} = e^{\mathbf{a}}_\alpha \nabla_{\mathbf{a}} e^\beta_{\mathbf{b}} + (\nabla_{\mathbf{b}} e^{\mathbf{a}}_\alpha) e^\beta_{\mathbf{a}} = -\Gamma^\beta_{\alpha\mathbf{b}} + \Gamma^\beta_{\mathbf{b}\alpha} = 0 \tag{1.2.41}$$

therefore

$$\mathcal{L}_{e_\alpha} g_{\mathbf{bc}} = \mathcal{L}_{e_\alpha} \left(e^\beta_{\mathbf{b}} e^\gamma_{\mathbf{c}} g_{\beta\gamma} \right) = e^\beta_{\mathbf{b}} e^\gamma_{\mathbf{c}} \nabla_\alpha g_{\beta\gamma} \tag{1.2.42}$$

hence a coordinate basis vector $e^{\mathbf{a}}_\alpha$ is a Killing vector if and only if the metric components are independent of the coordinate x^α.

If a particle with momentum

$$p_{\mathbf{a}} = m g_{\mathbf{ab}} \frac{dx^{\mathbf{b}}}{dt} \tag{1.2.43}$$

is moving under the influence of a force

$$f_{\mathbf{a}} = \frac{dp_{\mathbf{a}}}{dt} \tag{1.2.44}$$

in a space with Killing vector field $\xi^{\mathbf{a}}$ then, using Eq. (1.2.40),

$$\begin{aligned}
\frac{d}{dt}\left(p_{\mathbf{a}}\xi^{\mathbf{a}}\right) &= \frac{dp_{\mathbf{a}}}{dt}\xi^{\mathbf{a}} + p_{\mathbf{a}}\frac{d\xi^{\mathbf{a}}}{dt} && (1.2.45)\\
&= f_{\mathbf{a}}\xi^{\mathbf{a}} + mg_{\mathbf{ab}}\frac{dx^{\mathbf{b}}}{dt}\frac{dx^{\mathbf{c}}}{dt}\nabla_{\mathbf{c}}\xi^{\mathbf{a}} && (1.2.46)\\
&= f_{\mathbf{a}}\xi^{\mathbf{a}} + \frac{1}{2}m\frac{dx^{\mathbf{b}}}{dt}\frac{dx^{\mathbf{c}}}{dt}\left(\nabla_{\mathbf{b}}\xi_{\mathbf{c}} + \nabla_{\mathbf{c}}\xi_{\mathbf{b}}\right) && (1.2.47)\\
&= f_{\mathbf{a}}\xi^{\mathbf{a}} + \frac{1}{2}m\frac{dx^{\mathbf{b}}}{dt}\frac{dx^{\mathbf{c}}}{dt}\mathcal{L}_{\xi}g_{\mathbf{bc}} && (1.2.48)\\
&= f_{\mathbf{a}}\xi^{\mathbf{a}} && (1.2.49)
\end{aligned}$$

In particular, if the particle is moving freely then $p_{\mathbf{a}}\xi^{\mathbf{a}}$ is conserved. For example, from Eq. (1.2.27), two dimensional Euclidean space has translational symmetries generated by $e_x^{\mathbf{a}}$ and $e_y^{\mathbf{a}}$, which give rise to the conserved quantities

$$\begin{aligned}
p_{\mathbf{a}}e_x^{\mathbf{a}} &= p_x = mg_{xx}\dot{x} = m\dot{x} && (1.2.50)\\
p_{\mathbf{a}}e_y^{\mathbf{a}} &= p_y = mg_{yy}\dot{y} = m\dot{y} && (1.2.51)
\end{aligned}$$

and, from Eq. (1.2.28), has rotational symmetry generated by $e_\theta^{\mathbf{a}}$, which gives rise to the conserved quantity

$$p_{\mathbf{a}}e_\theta^{\mathbf{a}} = p_\theta = mg_{\theta\theta}\dot{\theta} = mr^2\dot{\theta} \tag{1.2.52}$$