Chapter 1

Differential Geometry

1.1 Tensors

A **tensor** is a mathematical object that directly represents a physical quantity. Tensors of the same type can be added, and multiplied by a scalar, in the usual way. Scalars and vectors are tensors, but many physical quantities are some other type of tensor.

A **scalar** is a tensor that behaves like a number. Examples of spatial[1] scalars are time t, energy E and electric potential ϕ. Examples of spacetime scalars are proper time τ, mass m and charge q.

1.1.1 Vectors and covectors

A **vector** is a tensor that behaves like an arrow. Their properties inspire the vector

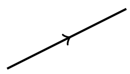

Figure 1.1.1: A vector.

space axioms of mathematics. A scalar times a vector is a vector and the sum of two vectors is a vector, see Figure 1.1.2. Examples of vectors are displacement $\vec{dx}$, velocity

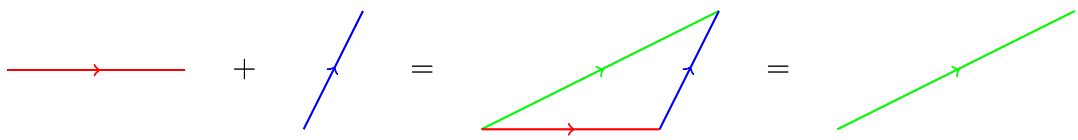

Figure 1.1.2: The sum of two vectors is a vector.

[1]Physical quantities may be one type of tensor with respect to one space but another type of tensor with respect to another space. For example, a displacement in time is a scalar with respect to space but a vector with respect to time. Unless otherwise specified, the space can be assumed to be space, or spacetime in the context of relativity.

$$\vec{v} \equiv \frac{d\vec{x}}{dt} \tag{1.1.1}$$

and acceleration

$$\vec{a} \equiv \frac{d\vec{v}}{dt} \tag{1.1.2}$$

A **covector** or **one-form** is a tensor that behaves like the local linearized form of contour lines or the crests of a wave, see Figure 1.1.3. A scalar times a covector

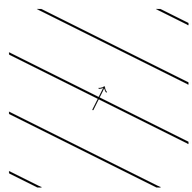

Figure 1.1.3: A covector.

is a covector and the sum of two covectors is a covector, see Figure 1.1.4. Examples

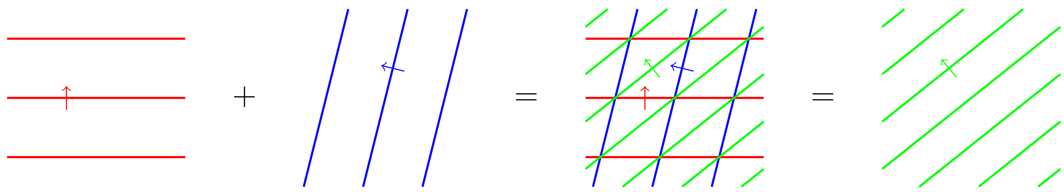

Figure 1.1.4: The sum of two covectors is a covector.

of covectors are the gradient of a scalar field $\underline{\nabla}\phi$, wave "vectors" $\underline{k}$, electric field and magnetic "vector" potential

$$\underline{E} = -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t} \tag{1.1.3}$$

momentum

$$\underline{p} = \hbar \underline{k} \tag{1.1.4}$$

and force

$$\underline{F} = q\underline{E} \tag{1.1.5}$$

or

$$\underline{F} = \frac{d\underline{p}}{dt} \tag{1.1.6}$$

A vector can be **contracted** with a covector to give a scalar

$$\vec{v} \cdot \underline{\omega} = \text{scalar} \tag{1.1.7}$$

corresponding to the number of covector planes crossed by the vector, with sign given by the relative orientations of the vector and covector, see Figure 1.1.5. For example, a

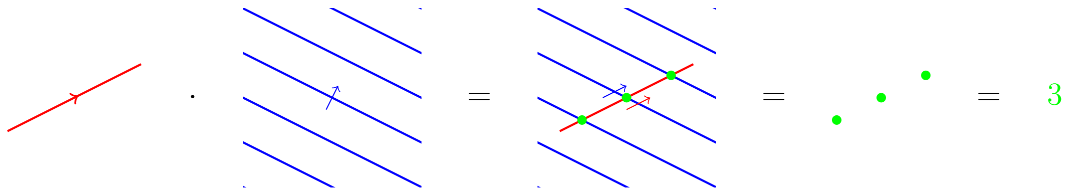

Figure 1.1.5: A vector contracted with a covector gives a scalar.

displacement contracted with the gradient of a scalar field gives the change in the scalar field

$$\vec{dx} \cdot \underline{\nabla}\phi = d\phi \tag{1.1.8}$$

and power equals force contracted with velocity

$$P = \underline{F} \cdot \vec{v} \tag{1.1.9}$$

Comparing vectors and covectors, the magnitude of a vector is given by its length, while the magnitude of a covector is given by the density of its planes. The direction of a vector is along its length (intrinsically oriented), while the direction of a covector is normal to its planes (extrinsically oriented) in the sense that $\underline{n} \cdot \vec{v} = 0$ for any vector $\vec{v}$ lying in the plane of the covector $\underline{n}$. Thus, a vector is an intrinsically oriented dimension one plane element, while a covector is an extrinsically oriented codimension[2] one plane density.

1.1.2 Abstract index notation

The notation $\vec{v}$ and $\underline{\omega}$ works well for vectors and covectors but is inadequate for more general tensors. Instead, we will use the **abstract index notation** in which a vector $\vec{v}$ is written $v^{\mathbf{a}}$ and a covector $\underline{\omega}$ is written $\omega_{\mathbf{a}}$

$$\vec{v} \;\leftrightarrow\; v^{\mathbf{a}} \quad , \quad \underline{\omega} \;\leftrightarrow\; \omega_{\mathbf{a}} \tag{1.1.10}$$

The abstract index **a** denotes the tensorial nature of the quantity by its position and does not take specific values. A contraction is denoted by repeated indices

$$\underline{\omega} \cdot \vec{v} \;\leftrightarrow\; \omega_{\mathbf{a}} v^{\mathbf{a}} \tag{1.1.11}$$

A **tensor** can have an arbitrary number of vector and covector indices $T^{\mathbf{ab}\cdots}_{\mathbf{cd}\cdots}$.

A tensor $T_{\mathbf{ab}}$ can be decomposed into **symmetric** and **antisymmetric** parts

$$T_{\mathbf{ab}} = T_{(\mathbf{ab})} + T_{[\mathbf{ab}]} \tag{1.1.12}$$

where round brackets denote the symmetric part

$$T_{(\mathbf{ab})} = \frac{1}{2}\left(T_{\mathbf{ab}} + T_{\mathbf{ba}}\right) \tag{1.1.13}$$

and square brackets denote the antisymmetric part

$$T_{[\mathbf{ab}]} = \frac{1}{2}\left(T_{\mathbf{ab}} - T_{\mathbf{ba}}\right) \tag{1.1.14}$$

[2] Codimension d is dimension $D - d$ where D is the dimension of the space.

1.1.3 Metric

The **metric** $g_{\mathbf{ab}}$ and inverse metric $g^{\mathbf{ab}}$ define lengths and angles in a space. They are symmetric tensors

$$g_{\mathbf{ab}} = g_{\mathbf{ba}} \quad , \quad g^{\mathbf{ab}} = g^{\mathbf{ba}} \tag{1.1.15}$$

and related by

$$g^{\mathbf{ab}} g_{\mathbf{bc}} = \delta^{\mathbf{a}}_{\mathbf{c}} \tag{1.1.16}$$

where the identity tensor $\delta^{\mathbf{a}}_{\mathbf{b}}$ has the property

$$\delta^{\mathbf{a}}_{\mathbf{b}} v^{\mathbf{b}} = v^{\mathbf{a}} \quad , \quad \delta^{\mathbf{b}}_{\mathbf{a}} \omega_{\mathbf{b}} = \omega_{\mathbf{a}} \tag{1.1.17}$$

and similarly for other tensors.

The metric gives the **inner product** of vectors and covectors

$$\vec{u} \cdot \vec{v} = g_{\mathbf{ab}} u^{\mathbf{a}} v^{\mathbf{b}} \quad , \quad \underline{\omega} \cdot \underline{\sigma} = g^{\mathbf{ab}} \omega_{\mathbf{a}} \sigma_{\mathbf{b}} \tag{1.1.18}$$

Note that these inner products depend on the metric, in contrast to the contraction of a vector with a covector.

The metric converts vectors into covectors and vice versa

$$\begin{aligned} v_{\mathbf{a}} &= g_{\mathbf{ab}} v^{\mathbf{b}} \quad , \quad v^{\mathbf{a}} = g^{\mathbf{ab}} v_{\mathbf{b}} \\ \omega^{\mathbf{a}} &= g^{\mathbf{ab}} \omega_{\mathbf{b}} \quad , \quad \omega_{\mathbf{a}} = g_{\mathbf{ab}} \omega^{\mathbf{b}} \end{aligned} \tag{1.1.19}$$

For example, the traditional vector representation $\vec{E}$ of the electric field $\underline{E}$ is

$$E^{\mathbf{a}} = g^{\mathbf{ab}} E_{\mathbf{b}} \tag{1.1.20}$$

More generally, the metric can **raise** or **lower indices** on any tensor [3]

$$T^{\mathbf{a}}{}_{\mathbf{b}} = g_{\mathbf{bc}} T^{\mathbf{ac}} \tag{1.1.21}$$

[3] It is important to maintain the horizontal position of the indices if the tensor is not symmetric since $T^{\mathbf{a}}{}_{\mathbf{b}} = g_{\mathbf{bc}} T^{\mathbf{ac}} \neq g_{\mathbf{bc}} T^{\mathbf{ca}} = T_{\mathbf{b}}{}^{\mathbf{a}}$ if $T^{\mathbf{ac}} \neq T^{\mathbf{ca}}$.