Homework 7 - Inflation

Q7.1. Calculate the power spectrum of curvature perturbations $P_{\mathcal{R}_c}(k)$ produced during inflation with scale factor $a \propto t^2$.

A7.1. Let

$$a = \alpha t^2 \tag{A7.1.1}$$

therefore

$$H = \frac{\dot{a}}{a} = \frac{2}{t} \tag{A7.1.2}$$

and

$$\dot{H} = -\frac{2}{t^2} \tag{A7.1.3}$$

Also, Eqs. (3.1.9), (144) and (145) give

$$\dot{H} = -\frac{1}{2}(\rho + p) = -\frac{1}{2}\dot{\phi}^2 \tag{A7.1.4}$$

therefore

$$\frac{a\dot{\phi}}{H} = \pm \alpha t^2 \tag{A7.1.5}$$

Now

$$\eta = \int \frac{dt}{a} = -\frac{1}{\alpha t} \tag{A7.1.6}$$

therefore

$$\frac{a\dot{\phi}}{H} = \pm \frac{1}{\alpha \eta^2} \tag{A7.1.7}$$

and

$$\frac{H}{a\dot{\phi}} \left(\frac{a\dot{\phi}}{H} \right)'' = \frac{6}{\eta^2} \tag{A7.1.8}$$

Therefore Eq. (203) becomes

$$\varphi_k'' + k^2 \varphi_k - \frac{6}{\eta^2} \varphi_k = 0 \tag{A7.1.9}$$

which from Eqs. (182) and (185) has solution

$$\varphi_k = -i\sqrt{\frac{\pi}{4k}}\sqrt{-k\eta}\, H^{(1)}_{5/2}(-k\eta) \tag{A7.1.10}$$

$$= \frac{1}{\sqrt{2k}} \left(1 - \frac{3i}{k\eta} - \frac{3}{k^2\eta^2} \right) e^{-ik\eta} \tag{A7.1.11}$$

$$\xrightarrow{k\eta \to 0} -\frac{3}{\sqrt{2k^5}\,\eta^2} \tag{A7.1.12}$$

Matching this with Eq. (212)

$$\varphi_k \stackrel{k\eta\to 0}{\longrightarrow} A_k \frac{a\dot{\phi}}{H} \tag{A7.1.13}$$

gives

$$A_k = -\frac{3}{\sqrt{2k^5}\,\eta^2}\frac{H}{a\dot{\phi}} = \mp\frac{3\alpha}{\sqrt{2k^5}} \tag{A7.1.14}$$

Therefore Eq. (220) gives

$$P_{\mathcal{R}_c}(k) = \frac{k^3}{2\pi^2}\left|A_k\right|^2 = \left(\frac{3\alpha}{2\pi k}\right)^2 = \left(\frac{3\alpha}{2\pi aH}\right)^2\Bigg|_{aH=k} = \left(\frac{3H}{8\pi}\right)^2\Bigg|_{aH=k} \tag{A7.1.15}$$

Note that

$$-\frac{\dot{H}}{H^2} = \frac{1}{2} \not\ll 1 \tag{A7.1.16}$$

and

$$\frac{\ddot{\phi}}{H\dot{\phi}} = -\frac{1}{2} \not\ll 1 \tag{A7.1.17}$$

so the slow-roll formula

$$P_{\mathcal{R}_c}(k) \simeq \left(\frac{H^2}{2\pi\dot{\phi}}\right)^2\Bigg|_{aH=k} = \left(\frac{H}{2\pi}\right)^2\Bigg|_{aH=k} \tag{A7.1.18}$$

is not accurate.