Homework 6 - Expanding universe

Q6.1. Calculate the time t as a function of the scale factor a for a universe dominated by radiation

$$p_\mathrm{r} = \frac{1}{3}\rho_\mathrm{r} \tag{Q6.1.1}$$

then matter

$$p_\mathrm{m} = 0 \tag{Q6.1.2}$$

then vacuum energy

$$p_\mathrm{v} = -\rho_\mathrm{v} \tag{Q6.1.3}$$

including the transitions between these eras, and show that your answer reduces to

$$a_\mathrm{r} \simeq \left(\frac{4\rho_\mathrm{r}a^4}{3}\right)^{\frac{1}{4}} t^{\frac{1}{2}} \tag{Q6.1.4}$$

$$a_\mathrm{m} \simeq \left(\frac{3\rho_\mathrm{m}a^3}{4}\right)^{\frac{1}{3}} t^{\frac{2}{3}} \tag{Q6.1.5}$$

$$a_\mathrm{v} \simeq \left(\frac{\rho_\mathrm{m}a^3}{4\rho_\mathrm{v}}\right)^{\frac{1}{3}} \exp\left(\sqrt{\frac{\rho_\mathrm{v}}{3}}\, t\right) \tag{Q6.1.6}$$

when the respective individual components dominate.

A6.1. Eqs. (3.1.7) and (3.1.10) and Eqs. (Q6.1.1), (Q6.1.2) and (Q6.1.3) give

$$\rho_\mathrm{r} \propto a^{-4} \tag{A6.1.1}$$

$$\rho_\mathrm{m} \propto a^{-3} \tag{A6.1.2}$$

$$\rho_\mathrm{v} \propto a^{0} \tag{A6.1.3}$$

and so Eq. (3.1.4) gives

$$3\left(\frac{\dot{a}}{a}\right)^2 = \rho = \left(\rho_\mathrm{r}a^4\right)a^{-4} + \left(\rho_\mathrm{m}a^3\right)a^{-3} + \rho_\mathrm{v} \tag{A6.1.4}$$

$$= Aa^{-4} + Ba^{-3} + C \tag{A6.1.5}$$

Therefore

$$\frac{1}{\sqrt{3}}\int_0^t dt = \int_0^a \frac{da}{a\sqrt{Aa^{-4}+Ba^{-3}+C}} \tag{A6.1.6}$$

$$= \int \frac{a\,da}{\sqrt{A+Ba+Ca^4}} \tag{A6.1.7}$$

$$\simeq \begin{cases} \displaystyle\int_0^a \frac{a\,da}{\sqrt{A+Ba}} & \text{for } a \le a_* \\ \displaystyle\int_0^{a_*} \frac{a\,da}{\sqrt{A+Ba}} + \int_{a_*}^a \frac{a\,da}{\sqrt{Ba+Ca^4}} & \text{for } a \ge a_* \end{cases} \tag{A6.1.8}$$

where

$$a_* \gg \frac{A}{B} \quad \Leftrightarrow \quad \rho_{\mathrm{m}}(a_*) \gg \rho_{\mathrm{r}}(a_*) \tag{A6.1.9}$$

$$a_*^3 \ll \frac{B}{C} \quad \Leftrightarrow \quad \rho_{\mathrm{m}}(a_*) \gg \rho_{\mathrm{v}} \tag{A6.1.10}$$

i.e. t_* is during matter domination.

Therefore for $a \leq a_*$, i.e. during radiation and matter domination,

$$\begin{aligned}
\frac{1}{\sqrt{3}} t &= \int_0^a \frac{a\,da}{\sqrt{A+Ba}} && \text{(A6.1.11)}\\
&= \frac{1}{B}\int_0^a \frac{(A+Ba-A)\,da}{\sqrt{A+Ba}} && \text{(A6.1.12)}\\
&= \frac{2}{3B^2}\left[(A+Ba)^{3/2} - 3A\,(A+Ba)^{1/2} + 2A^{3/2}\right] && \text{(A6.1.13)}\\
&= \frac{2}{3B^2}\left(\sqrt{A+Ba} + 2\sqrt{A}\right)\left(\sqrt{A+Ba} - \sqrt{A}\right)^2 && \text{(A6.1.14)}\\
&\sim \begin{cases} \dfrac{a^2}{2A^{1/2}} & \text{for } a \ll \dfrac{A}{B} \quad \Leftrightarrow \quad \rho_{\mathrm{r}} \gg \rho_{\mathrm{m}} \\[2ex] \dfrac{2a^{3/2}}{3B^{1/2}} & \text{for } a \gg \dfrac{A}{B} \quad \Leftrightarrow \quad \rho_{\mathrm{m}} \gg \rho_{\mathrm{r}} \end{cases} && \text{(A6.1.15)}
\end{aligned}$$

or

$$a \sim \begin{cases} \left(\dfrac{4A}{3}\right)^{\frac{1}{4}} t^{\frac{1}{2}} & \text{for } a \ll \dfrac{A}{B} \quad \Leftrightarrow \quad \rho_{\mathrm{r}} \gg \rho_{\mathrm{m}} \\[2ex] \left(\dfrac{3}{4}B\right)^{\frac{1}{3}} t^{\frac{2}{3}} & \text{for } a \gg \dfrac{A}{B} \quad \Leftrightarrow \quad \rho_{\mathrm{m}} \gg \rho_{\mathrm{r}} \end{cases} \tag{A6.1.16}$$

Also for $a \geq a_*$, i.e. during matter and vacuum domination,

$$\begin{aligned}
\frac{1}{\sqrt{3}}(t - t_*) &= \int_{a_*}^a \frac{a\,da}{\sqrt{Ba + Ca^4}} && \text{(A6.1.17)}\\
&= \frac{1}{3}\int_{a_*}^a \frac{3a^2\,da}{\sqrt{Ba^3 + Ca^6}} && \text{(A6.1.18)}\\
&= \frac{1}{3\sqrt{C}}\int_{a_*}^a \frac{3a^2\,da}{\sqrt{\left(\frac{B}{2C} + a^3\right)^2 - \left(\frac{B}{2C}\right)^2}} && \text{(A6.1.19)}\\
&= \frac{1}{3\sqrt{C}}\left[\cosh^{-1}\left(1 + \frac{2C}{B}a^3\right) - \cosh^{-1}\left(1 + \frac{2C}{B}a_*^3\right)\right] && \text{(A6.1.20)}\\
&\simeq \frac{1}{3\sqrt{C}}\cosh^{-1}\left(1 + \frac{2C}{B}a^3\right) - \frac{2a_*^{3/2}}{3B^{1/2}} && \text{(A6.1.21)}
\end{aligned}$$

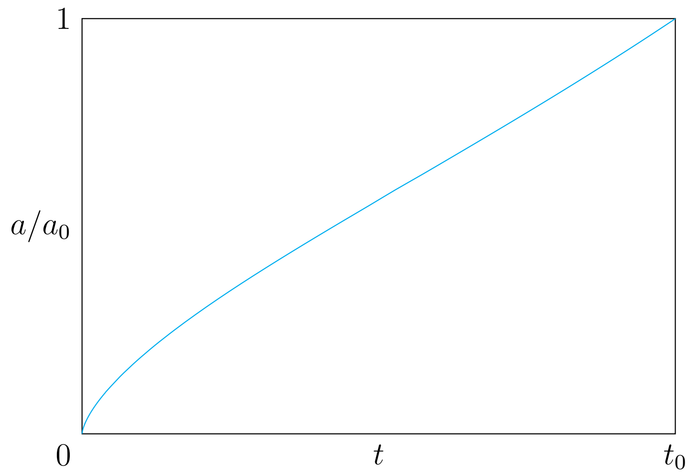

Figure A6.1.1: The expansion of the universe is just beginning to accelerate.

therefore

$$\frac{1}{\sqrt{3}}t = \frac{1}{3\sqrt{C}}\cosh^{-1}\left(1+\frac{2C}{B}a^3\right) \tag{A6.1.22}$$

$$\sim \begin{cases} \dfrac{2a^{3/2}}{3B^{1/2}} & \text{for } a^3 \ll \dfrac{B}{C} \Leftrightarrow \rho_{\mathrm{m}} \gg \rho_{\mathrm{v}} \\ \dfrac{1}{3\sqrt{C}}\ln\left(\dfrac{4C}{B}a^3\right) & \text{for } a^3 \gg \dfrac{B}{C} \Leftrightarrow \rho_{\mathrm{v}} \gg \rho_{\mathrm{m}} \end{cases} \tag{A6.1.23}$$

or

$$a = \left\{\frac{B}{2C}\left[\cosh\left(\sqrt{3C}\,t\right)-1\right]\right\}^{\frac{1}{3}} \tag{A6.1.24}$$

$$\sim \begin{cases} \left(\dfrac{3}{4}Bt^2\right)^{\frac{1}{3}} & \text{for } a^3 \ll \dfrac{B}{C} \Leftrightarrow \rho_{\mathrm{m}} \gg \rho_{\mathrm{v}} \\ \left(\dfrac{B}{4C}\right)^{\frac{1}{3}}\exp\left(\sqrt{\dfrac{C}{3}}\,t\right) & \text{for } a^3 \gg \dfrac{B}{C} \Leftrightarrow \rho_{\mathrm{v}} \gg \rho_{\mathrm{m}} \end{cases} \tag{A6.1.25}$$

Combining Eqs. (A6.1.14) and (A6.1.22) gives

$$t = \begin{cases} \dfrac{2}{\sqrt{3}\,\rho_{\mathrm{m}}^2}\left(\sqrt{\rho_{\mathrm{r}}+\rho_{\mathrm{m}}}+2\sqrt{\rho_{\mathrm{r}}}\right)\left(\sqrt{\rho_{\mathrm{r}}+\rho_{\mathrm{m}}}-\sqrt{\rho_{\mathrm{r}}}\right)^2 & \text{for } \rho_{\mathrm{r}}+\rho_{\mathrm{m}} \gg \rho_{\mathrm{v}} \\ \dfrac{1}{\sqrt{3\rho_{\mathrm{v}}}}\cosh^{-1}\left(1+\dfrac{2\rho_{\mathrm{v}}}{\rho_{\mathrm{m}}}\right) & \text{for } \rho_{\mathrm{m}}+\rho_{\mathrm{v}} \gg \rho_{\mathrm{r}} \end{cases} \tag{A6.1.26}$$

during radiation and matter domination and matter and vacuum domination, respectively, and Eqs. (A6.1.16) and (A6.1.25) give Eqs. (Q6.1.4), (Q6.1.5) and (Q6.1.6). See Figure A6.1.1.