Homework 4 - Schwarzschild metric

Q4.1. You synchronize two clocks and then throw one up into the air and catch it. Which clock will show the earlier time? By how much?

A4.1. Extremizing the action for a free particle

$$S = -m \int d\tau \tag{A4.1.1}$$

see Eq. (2.3.5), maximizes the length (proper time interval) of the worldline, since null deformations reduce the length of a timelike curve. Thus the freely moving clock, i.e. the thrown clock, will show the later time and the held clock will show the earlier time.

We can see how this works in Schwarzschild coordinates. For radial motion, Eq (2.6.2) gives

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{2GM}{r} - \left(1 - \frac{2GM}{r}\right)^{-1} \left(\frac{dr}{dt}\right)^2} \tag{A4.1.2}$$

Taking $r \simeq r_0 \gg GM$ and using the Newtonian solution of Eqs. (2.6.9) and (2.6.13)

$$r \simeq r_0 + v_0 t - \frac{1}{2} g_0 t^2 \tag{A4.1.3}$$

with

$$g_0 = \frac{GM}{r_0^2} \tag{A4.1.4}$$

gives

$$\frac{d\tau}{dt} \simeq 1 - r_0 g_0 + g_0 \left(v_0 t - \frac{1}{2} g_0 t^2\right) - \frac{1}{2} \left(v_0 - g_0 t\right)^2 \tag{A4.1.5}$$

Integrating and evaluating when the clock returns at $t \simeq 2v_0/g_0$ gives

$$\tau_{\text{thrown}} \simeq \frac{2v_0}{g_0} - 2r_0 v_0 + \frac{2}{3} \frac{v_0^3}{g_0} - \frac{1}{6} \frac{v_0^3}{g_0} \tag{A4.1.6}$$

for the thrown clock compared with

$$\tau_{\text{held}} \simeq \frac{2v_0}{g_0} - 2r_0 v_0 \tag{A4.1.7}$$

for the held clock. Thus the reduction in the gravitional time dilation for the thrown clock is four times its velocity time dilation and so the held clock shows a time $v_0^3/2g_0$ earlier than the thrown clock.

Q4.2. Consider the spacetime with metric

$$d\tau^2 = \left(1 - \frac{2GM}{r} - H^2 r^2\right) dt^2 - \left(1 - \frac{2GM}{r} - H^2 r^2\right)^{-1} dr^2 - r^2\, d\Omega^2 \tag{Q4.2.1}$$

(a) Describe the properties of the spacetime.

(b) Draw its Penrose diagram.

(c) Describe the possible circular orbits of a particle moving freely in this spacetime.

A4.2. (a) The spacetime is static and spherically symmetric, and has a physical singularity at $r = 0$.

g_{tt} is zero and g_{rr} diverges when

$$g_{tt} = 1 - \frac{2GM}{r} - H^2 r^2 = 0 \tag{A4.2.1}$$

which has two solutions if

$$GMH < \frac{1}{3\sqrt{3}} \tag{A4.2.2}$$

corresponding to an event horizon at $r = r_{\rm e}$ and a cosmological horizon at $r = r_{\rm c}$, with

$$2GM < r_{\rm e} < 3GM < \frac{1}{\sqrt{3}\,H} < r_{\rm c} < \frac{1}{H} \tag{A4.2.3}$$

The case $3\sqrt{3}\,GMH \geq 1$ is unphysical corresponding to a naked singularity.

(b) The Penrose diagram is a combination of the Schwarzschild and de Sitter Penrose diagrams, see Figure A4.2.1.

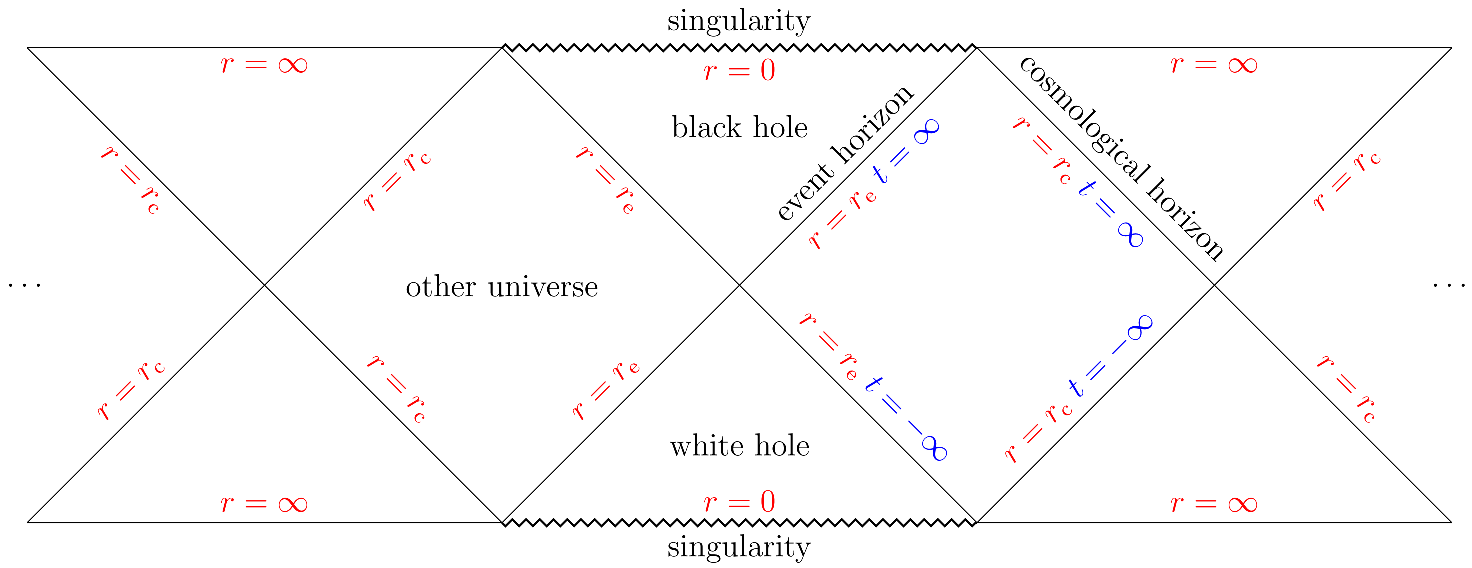

Figure A4.2.1: Penrose diagram for Schwarzschild de Sitter spacetime. The pattern repeats indefinitely.

(c) Following Section 2.6.1, fixing $\theta = \pi/2$ and using the results of Section 1.2.3, time translational and rotational symmetries give the conserved quantities

$$E = e_t^{\mathbf{a}} p_{\mathbf{a}} = m g_{t\mathbf{b}} \frac{dx^{\mathbf{b}}}{d\tau} = m \left(1 - \frac{2GM}{r} - H^2 r^2\right) \frac{dt}{d\tau} \tag{A4.2.4}$$

and

$$L = -e_\phi^{\mathbf{a}} p_{\mathbf{a}} = -m g_{\phi\mathbf{b}} \frac{dx^{\mathbf{b}}}{d\tau} = m r^2 \frac{d\phi}{d\tau} \tag{A4.2.5}$$

while the radial motion can be determined using

$$m^2 = g^{\mathbf{ab}} p_{\mathbf{a}} p_{\mathbf{b}} = g^{tt} p_t^2 + g^{rr} p_r^2 + g^{\phi\phi} p_\phi^2 \tag{A4.2.6}$$

$$= \left(1 - \frac{2GM}{r} - H^2 r^2\right)^{-1} \left[E^2 - m^2 \left(\frac{dr}{d\tau}\right)^2\right] - \frac{L^2}{r^2} \tag{A4.2.7}$$

therefore

$$\frac{1}{2} m \left(\frac{dr}{d\tau}\right)^2 + V(r) = \frac{E^2 - m^2}{2m} \tag{A4.2.8}$$

where the effective potential

$$V(r) = -\frac{GML^2}{mr^3} + \frac{L^2}{2mr^2} - \frac{GMm}{r} - \frac{H^2 L^2}{2m} - \frac{1}{2} m H^2 r^2 \tag{A4.2.9}$$

and

$$V'(r) = \frac{3GML^2}{mr^4} - \frac{L^2}{mr^3} + \frac{GMm}{r^2} - mH^2 r \tag{A4.2.10}$$

In general, the effective potential will have two maxima and a minimum, the first maximum at

$$r \sim 3GM \left[1 + 3\left(\frac{GMm}{L}\right)^2 + \ldots\right] \tag{A4.2.11}$$

corresponding to the Schwarzschild unstable circular orbit, the minimum at

$$r \sim \frac{L^2}{GMm^2} \tag{A4.2.12}$$

to the Newtonian stable circular orbit, and the second maximum at

$$r \sim \left(\frac{GM}{H^2}\right)^{\frac{1}{3}} - \frac{L^2}{3GMm^2} + \ldots \tag{A4.2.13}$$

to a de Sitter unstable circular orbit balancing the Newtonian attraction against the de Sitter repulsion. If

$$L \lesssim 2\sqrt{3}\, GMm \tag{A4.2.14}$$

then the Schwarzschild and Newtonian orbits disappear leaving the de Sitter unstable circular orbit. If

$$L \gtrsim \frac{\sqrt{3}\,m}{2}\left(\frac{G^2M^2}{2H}\right)^{\frac{1}{3}} \tag{A4.2.15}$$

then the de Sitter and Newtonian orbits disappear leaving the Schwarzschild unstable circular orbit.

Q4.3. A particle of mass m falls towards a Schwarzschild black hole of mass M. The particle is initially a great distance from the black hole and has energy $E = m$ and angular momentum $L \simeq 4GMm$.

(a) In the case $L > 4GMm$, describe qualitatively its trajectory.

(b) In the case $L < 4GMm$, describe qualitatively its trajectory.

(c) In the case $L = 4GMm$, calculate its trajectory $r(\phi)$ and asymptotic orbital period from the point of view of

i. an astronaut following the trajectory

ii. a distant observer

A4.3. For $E = m$ and $L = 4GMm$, Eqs. (2.6.13) and (2.6.14) become

$$\frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 + V(r) = 0 \tag{A4.3.1}$$

with

$$V(r) = -\frac{GMm}{r}\left(1 - \frac{4GM}{r}\right)^2 \tag{A4.3.2}$$

$r = 4GM$ corresponds to an unstable circular orbit and, since $V(4GM) = V(\infty)$, a particle falling in from infinity will asymptotically approach this orbit.

(a) In the case of L slightly greater than $4GMm$, the angular momentum barrier will be slightly higher and the particle will approach $r = 4GM$, orbiting there for some period of time, before eventually escaping back to infinity.

(b) In the case of L slightly less than $4GMm$, the angular momentum barrier will be slightly lower and the particle will approach $r = 4GM$, orbiting there for some period of time, before eventually falling into the black hole.

(c) For $E = m$ and $L = 4GMm$, Eqs. (2.6.9) and (2.6.10) become

$$\frac{dt}{d\tau} = \left(1 - \frac{2GM}{r}\right)^{-1} \tag{A4.3.3}$$

and

$$\frac{d\phi}{d\tau} = \frac{4GM}{r^2} \tag{A4.3.4}$$

Combining Eqs. (A4.3.1) and (A4.3.4) gives

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^3}{8GM}\left(1 - \frac{4GM}{r}\right)^2 \tag{A4.3.5}$$

Using the change of variables

$$y^2 = x = \frac{r}{4GM} \tag{A4.3.6}$$

integrating Eq. (A4.3.5) gives

$$\begin{aligned}
\frac{1}{\sqrt{2}}\int d\phi &= -\int \frac{dr}{4GM\sqrt{\left(\frac{r}{4GM}\right)^3\left(1-\frac{4GM}{r}\right)^2}} && \text{(A4.3.7)}\\
&= -\int \frac{x^{-1/2}\,dx}{x-1} && \text{(A4.3.8)}\\
&= -\int \frac{2\,dy}{y^2-1} && \text{(A4.3.9)}\\
&= 2\coth^{-1} y && \text{(A4.3.10)}
\end{aligned}$$

Therefore, setting $\phi = 0$ at $r \to \infty$, the particle's trajectory is

$$r = 4GM\coth^2\left(\frac{\phi}{2\sqrt{2}}\right) \tag{A4.3.11}$$

The trajectory asymptotically approaches the circular orbit at $r = 4GM$, where Eqs. (A4.3.3) and (A4.3.4) reduce to

$$\frac{dt}{d\tau} = 2 \tag{A4.3.12}$$

and

$$\frac{d\phi}{d\tau} = \frac{1}{4GM} \tag{A4.3.13}$$

Therefore the asymptotic periods are

i. $8\pi GM$ for an astronaut following the trajectory

ii. $16\pi GM$ for a distant observer