Homework 3 - Minkowski space

Q3.1. Determine the nature of the spacetime with metric

$$d\tau^2 = dt^2 - t^2 \left(d\xi^2 + \sinh^2\xi \, d\Omega^2\right) \tag{Q3.1.1}$$

A3.1. The spacetime appears to be an expanding universe with scale factor $a(t) = t$ and negatively curved spatial hypersurfaces. However, the metric of Eq. (Q3.1.1) can be obtained as the $H \to 0$ limit of the de Sitter metric Eq. (2.5.5)

$$d\tau^2 = dt^2 - H^{-2}\sinh^2(Ht) \left(d\xi^2 + \sinh^2\xi \, d\Omega^2\right) \tag{A3.1.1}$$

and so corresponds to the $H \to 0$ limit of de Sitter space, i.e. Minkowski space. For example, the $H \to 0$ limit of the de Sitter metric Eq. (2.5.4)

$$d\tau^2 = dt'^2 - \exp(2Ht') \left(dr^2 + r^2 \, d\Omega^2\right) \tag{A3.1.2}$$

or Eq. (2.5.6)

$$d\tau^2 = \left(1 - H^2 r^2\right) dt'^2 - \left(\frac{dr^2}{1 - H^2 r^2} + r^2 \, d\Omega^2\right) \tag{A3.1.3}$$

is the Minkowski metric

$$d\tau^2 = dt'^2 - dr^2 - r^2 \, d\Omega^2 \tag{A3.1.4}$$

More directly, the change in coordinates

$$t' = t\cosh\xi \tag{A3.1.5}$$

$$r = t\sinh\xi \tag{A3.1.6}$$

transforms the Minkowski metric Eq. (A3.1.4) into the metric of Eq. (Q3.1.1) and so both describe the same spacetime, Minkowski space.

Q3.2. Draw the penrose diagram for Minkowski space.

A3.2. The Minkowski metric is

$$d\tau^2 = dt^2 - dr^2 - r^2 \, d\Omega^2 \tag{A3.2.1}$$

Defining the null coordinates

$$u = (t + r)/\sqrt{2} \tag{A3.2.2}$$

$$v = (t - r)/\sqrt{2} \tag{A3.2.3}$$

gives

$$d\tau^2 = 2\, du\, dv - \frac{1}{2}(u - v)^2 \, d\Omega^2 \tag{A3.2.4}$$

The coordinates

$$u' = \tan^{-1} u \quad \text{(A3.2.5)}$$
$$v' = \tan^{-1} v \quad \text{(A3.2.6)}$$

bring infinity in to finite values

$$u', v' \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \quad \text{(A3.2.7)}$$

and a conformal rescaling of the metric

$$\hat{g}_{\mathbf{ab}} = \cos^2 u' \cos^2 v' g_{\mathbf{ab}} \quad \text{(A3.2.8)}$$

gives

$$d\hat{\tau}^2 = 2\, du'\, dv' - \frac{1}{2} \sin^2(u' - v')\, d\Omega^2 \quad \text{(A3.2.9)}$$

and hence Figure A3.2.1.

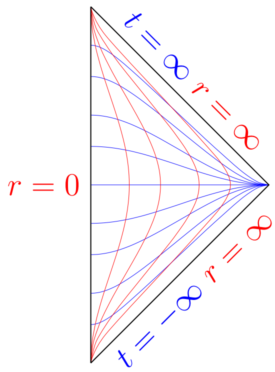

Figure A3.2.1: Penrose diagram for Minkowski spacetime.