

## Homework 1 - Newtonian perspective

Q1.1. For a particle with spacetime velocity  $u^{\mathbf{a}}$  experiencing a spacetime force  $f_{\mathbf{a}}$ , show that

$$f_{\mathbf{a}}u^{\mathbf{a}} = 0 \quad (\text{Q1.1.1})$$

and interpret this equation from a Newtonian perspective.

A1.1. Using Eqs. (2.3.13) and (2.3.3),

$$f_{\mathbf{a}}u^{\mathbf{a}} = \frac{dp_{\mathbf{a}}}{d\tau}u^{\mathbf{a}} = mg_{\mathbf{ab}}\frac{du^{\mathbf{b}}}{d\tau}u^{\mathbf{a}} = \frac{1}{2}m\frac{d}{d\tau}(g_{\mathbf{ab}}u^{\mathbf{a}}u^{\mathbf{b}}) = 0 \quad (\text{A1.1.1})$$

Decomposing as in Eqs. (2.3.16) and (2.3.18),

$$0 = f_{\mathbf{a}}u^{\mathbf{a}} \quad (\text{A1.1.2})$$

$$= \frac{dt}{d\tau}(Pe_{\mathbf{a}}^t - F_{\mathbf{a}})\frac{dt}{d\tau}(e_t^{\mathbf{a}} + v^{\mathbf{a}}) \quad (\text{A1.1.3})$$

$$= \left(\frac{dt}{d\tau}\right)^2(P - F_{\mathbf{a}}v^{\mathbf{a}}) \quad (\text{A1.1.4})$$

Thus Eq. (Q1.1.1) corresponds to the Newtonian work-energy theorem.

Q1.2. In the space-time decomposition of the metric, Eq. (2.2.9), set

$$A = 1 + 2\phi \quad (\text{Q1.2.1})$$

and use the Newtonian approximation

$$\phi, B_{\mathbf{a}}v^{\mathbf{a}}, h_{\mathbf{ab}}v^{\mathbf{a}}v^{\mathbf{b}} \ll 1 \quad (\text{Q1.2.2})$$

- Reexpress the relativistic uncharged particle action in space-time decomposed form and interpret the leading terms in your action.
- Calculate the Euler-Lagrange equation for the leading terms in your action and interpret the resulting equation.
- Reexpress the spacetime momentum in space-time decomposed form and interpret the leading terms.

A1.2. (a) Using Eqs. (2.3.10), (2.2.9), (2.3.16) and (Q1.2.1), in the Newtonian limit, the relativistic particle action is

$$-S = m \int \sqrt{g_{\mathbf{ab}}\dot{x}^{\mathbf{a}}\dot{x}^{\mathbf{b}}} dt \quad (\text{A1.2.1})$$

$$= m \int \sqrt{(Ae_{\mathbf{a}}^t e_{\mathbf{b}}^t - B_{\mathbf{a}}e_{\mathbf{b}}^t - e_{\mathbf{a}}^t B_{\mathbf{b}} - h_{\mathbf{ab}})(e_t^{\mathbf{a}} + v^{\mathbf{a}})(e_t^{\mathbf{b}} + v^{\mathbf{b}})} dt \quad (\text{A1.2.2})$$

$$= m \int \sqrt{1 + 2\phi - 2B_{\mathbf{a}}v^{\mathbf{a}} - h_{\mathbf{ab}}v^{\mathbf{a}}v^{\mathbf{b}}} dt \quad (\text{A1.2.3})$$

$$\simeq m \int \left(1 + \phi - B_{\mathbf{a}}v^{\mathbf{a}} - \frac{1}{2}h_{\mathbf{ab}}v^{\mathbf{a}}v^{\mathbf{b}}\right) dt \quad (\text{A1.2.4})$$

We can interpret  $\phi$  as the Newtonian gravitational potential,  $B_{\mathbf{a}}$  as the gravitational analog of the magnetic covector potential  $A_{\mathbf{a}}^{(3)}$  and  $\frac{1}{2}mh_{\mathbf{ab}}v^{\mathbf{a}}v^{\mathbf{b}}$  as the Newtonian kinetic energy.

- (b) For the action of Eq. (A1.2.4), the Euler-Lagrange equation

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_3^{\mathbf{a}}} \right) = \frac{\partial L}{\partial x_3^{\mathbf{a}}} \quad (\text{A1.2.5})$$

gives

$$m \frac{d}{dt} (h_{\mathbf{ab}}v^{\mathbf{b}} + B_{\mathbf{a}}) = -m \nabla_{\mathbf{a}}^{(3)} \phi + mv^{\mathbf{b}} \nabla_{\mathbf{a}}^{(3)} B_{\mathbf{b}} \quad (\text{A1.2.6})$$

therefore

$$m \frac{dv_{\mathbf{a}}}{dt} = m \left[ \left( -\nabla_{\mathbf{a}}^{(3)} \phi - \dot{B}_{\mathbf{a}} \right) + \left( \nabla_{\mathbf{a}}^{(3)} B_{\mathbf{b}} - \nabla_{\mathbf{b}}^{(3)} B_{\mathbf{a}} \right) v^{\mathbf{b}} \right] \quad (\text{A1.2.7})$$

We can interpret  $-\nabla_{\mathbf{a}}^{(3)} \phi$  as the Newtonian gravitational field analogous to the electrostatic field,  $-\nabla_{\mathbf{a}}^{(3)} \phi - \dot{B}_{\mathbf{a}}$  as the gravitational analog of the electric field  $E_{\mathbf{a}} = -\nabla_{\mathbf{a}}^{(3)} \phi - \dot{A}_{\mathbf{a}}^{(3)}$ , and  $\nabla_{\mathbf{a}}^{(3)} B_{\mathbf{b}} - \nabla_{\mathbf{b}}^{(3)} B_{\mathbf{a}}$  as the gravitational analog of the magnetic flux density  $B_{\mathbf{ab}} = \nabla_{\mathbf{a}}^{(3)} A_{\mathbf{b}}^{(3)} - \nabla_{\mathbf{b}}^{(3)} A_{\mathbf{a}}^{(3)}$ . The whole equation is thus the gravitational analog of the Lorentz force law Eq. (2.3.26).

- (c) The spacetime momentum is decomposed in terms of the energy and spatial momentum as

$$p_{\mathbf{a}} = E e_{\mathbf{a}}^t - p_{\mathbf{a}}^{(3)} \quad (\text{A1.2.8})$$

Substituting Eqs. (2.2.9), (2.3.16) and (Q1.2.1) into Eq. (2.3.13) gives

$$p_{\mathbf{a}} = m g_{\mathbf{ab}} \frac{dx^{\mathbf{b}}}{d\tau} \quad (\text{A1.2.9})$$

$$= m \frac{dt}{d\tau} (A e_{\mathbf{a}}^t e_{\mathbf{b}}^t - B_{\mathbf{a}} e_{\mathbf{b}}^t - e_{\mathbf{a}}^t B_{\mathbf{b}} - h_{\mathbf{ab}}) (e_t^{\mathbf{b}} + v^{\mathbf{b}}) \quad (\text{A1.2.10})$$

$$= m \frac{dt}{d\tau} \left[ (1 + 2\phi - B_{\mathbf{b}}v^{\mathbf{b}}) e_{\mathbf{a}}^t - (B_{\mathbf{a}} + h_{\mathbf{ab}}v^{\mathbf{b}}) \right] \quad (\text{A1.2.11})$$

and, from Eq. (A1.2.3),

$$\frac{d\tau}{dt} = \sqrt{1 + 2\phi - 2B_{\mathbf{a}}v^{\mathbf{a}} - h_{\mathbf{ab}}v^{\mathbf{a}}v^{\mathbf{b}}} \quad (\text{A1.2.12})$$

Therefore, comparing Eqs. (A1.2.8) and (A1.2.11), in the Newtonian limit,

$$E = \frac{m(1 + 2\phi - B_{\mathbf{a}}v^{\mathbf{a}})}{\sqrt{1 + 2\phi - 2B_{\mathbf{a}}v^{\mathbf{a}} - h_{\mathbf{ab}}v^{\mathbf{a}}v^{\mathbf{b}}}} \quad (\text{A1.2.13})$$

$$\simeq m + m\phi + \frac{1}{2}mh_{\mathbf{ab}}v^{\mathbf{a}}v^{\mathbf{b}} \quad (\text{A1.2.14})$$

We can interpret  $m$  as the mass energy,  $m\phi$  as the Newtonian gravitational potential energy and  $\frac{1}{2}mh_{\mathbf{ab}}v^{\mathbf{a}}v^{\mathbf{b}}$  as the Newtonian kinetic energy. Also

$$p_{\mathbf{a}}^{(3)} = \frac{m(B_{\mathbf{a}} + h_{\mathbf{ab}}v^{\mathbf{b}})}{\sqrt{1 + 2\phi - 2B_{\mathbf{a}}v^{\mathbf{a}} - h_{\mathbf{ab}}v^{\mathbf{a}}v^{\mathbf{b}}}} \quad (\text{A1.2.15})$$

$$\simeq mh_{\mathbf{ab}}v^{\mathbf{b}} + mB_{\mathbf{a}} \quad (\text{A1.2.16})$$

We can interpret  $mh_{\mathbf{ab}}v^{\mathbf{b}}$  as the Newtonian momentum and  $mB_{\mathbf{a}}$  as the gravitational momentum analogous to the electromagnetic momentum  $qA_{\mathbf{a}}^{(3)}$ .