## Final Exam - 1pm to 3:45pm Tuesday 30th June, Online

Your answers should be clear and concise. They should start from basic principles and proceed logically. You may cite material from the lecture notes and homework answers.

Q1. Show Eqs. (2.2.13) and (2.2.22).

A1.

$$
\begin{aligned}
g_{\mathbf{ab}}g^{\mathbf{bc}} &= \left(Ae^t_{\mathbf{a}}e^t_{\mathbf{b}} - B_{\mathbf{a}}e^t_{\mathbf{b}} - e^t_{\mathbf{a}}B_{\mathbf{b}} - h_{\mathbf{ab}}\right)\left[\frac{\left(e^{\mathbf{b}}_t - B^{\mathbf{b}}\right)\left(e^{\mathbf{c}}_t - B^{\mathbf{c}}\right)}{A + B^{\mathbf{d}}B_{\mathbf{d}}} - h^{\mathbf{bc}}\right] && \text{(A1.1)} \\
&= \frac{Ae^t_{\mathbf{a}}e^t_{\mathbf{b}}e^{\mathbf{b}}_t\left(e^{\mathbf{c}}_t - B^{\mathbf{c}}\right)}{A + B^{\mathbf{d}}B_{\mathbf{d}}} - \frac{B_{\mathbf{a}}e^t_{\mathbf{b}}e^{\mathbf{b}}_t\left(e^{\mathbf{c}}_t - B^{\mathbf{c}}\right)}{A + B^{\mathbf{d}}B_{\mathbf{d}}} + \frac{e^t_{\mathbf{a}}B_{\mathbf{b}}B^{\mathbf{b}}\left(e^{\mathbf{c}}_t - B^{\mathbf{c}}\right)}{A + B^{\mathbf{d}}B_{\mathbf{d}}} \\
&\quad + e^t_{\mathbf{a}}B_{\mathbf{b}}h^{\mathbf{bc}} + \frac{h_{\mathbf{ab}}B^{\mathbf{b}}\left(e^{\mathbf{c}}_t - B^{\mathbf{c}}\right)}{A + B^{\mathbf{d}}B_{\mathbf{d}}} + h_{\mathbf{ab}}h^{\mathbf{bc}} && \text{(A1.2)} \\
&= \frac{Ae^t_{\mathbf{a}}\left(e^{\mathbf{c}}_t - B^{\mathbf{c}}\right)}{A + B^{\mathbf{d}}B_{\mathbf{d}}} - \frac{B_{\mathbf{a}}\left(e^{\mathbf{c}}_t - B^{\mathbf{c}}\right)}{A + B^{\mathbf{d}}B_{\mathbf{d}}} + \frac{e^t_{\mathbf{a}}B_{\mathbf{b}}B^{\mathbf{b}}\left(e^{\mathbf{c}}_t - B^{\mathbf{c}}\right)}{A + B^{\mathbf{d}}B_{\mathbf{d}}} \\
&\quad + e^t_{\mathbf{a}}B^{\mathbf{c}} + \frac{B_{\mathbf{a}}\left(e^{\mathbf{c}}_t - B^{\mathbf{c}}\right)}{A + B^{\mathbf{d}}B_{\mathbf{d}}} + h_{\mathbf{ab}}h^{\mathbf{bc}} && \text{(A1.3)} \\
&= e^t_{\mathbf{a}}\left(e^{\mathbf{c}}_t - B^{\mathbf{c}}\right) + e^t_{\mathbf{a}}B^{\mathbf{c}} + \delta^{\mathbf{c}}_{\mathbf{a}} - e^t_{\mathbf{a}}e^{\mathbf{c}}_t && \text{(A1.4)} \\
&= \delta^{\mathbf{c}}_{\mathbf{a}} && \text{(A1.5)}
\end{aligned}
$$

$$
\begin{aligned}
g^{\mathbf{ab}}e^t_{\mathbf{b}} &= \left[\frac{\left(e^{\mathbf{a}}_t - B^{\mathbf{a}}\right)\left(e^{\mathbf{b}}_t - B^{\mathbf{b}}\right)}{A + B^{\mathbf{c}}B_{\mathbf{c}}} - h^{\mathbf{ab}}\right]e^t_{\mathbf{b}} && \text{(A1.6)} \\
&= \frac{\left(e^{\mathbf{a}}_t - B^{\mathbf{a}}\right)e^{\mathbf{b}}_t e^t_{\mathbf{b}}}{A + B^{\mathbf{c}}B_{\mathbf{c}}} && \text{(A1.7)} \\
&= \frac{e^{\mathbf{a}}_t - B^{\mathbf{a}}}{A + B^{\mathbf{c}}B_{\mathbf{c}}} && \text{(A1.8)}
\end{aligned}
$$

and

$$g^{\mathbf{ab}}e^t_{\mathbf{a}}e^t_{\mathbf{b}} = \frac{1}{A + B^{\mathbf{c}}B_{\mathbf{c}}} \tag{A1.9}$$

therefore

$$n^{\mathbf{a}} = \frac{g^{\mathbf{ab}}e^t_{\mathbf{b}}}{g^{\mathbf{cd}}e^t_{\mathbf{c}}e^t_{\mathbf{d}}} = e^{\mathbf{a}}_t - B^{\mathbf{a}} \tag{A1.10}$$

hence

$$e^{\mathbf{a}}_t = n^{\mathbf{a}} + B^{\mathbf{a}} \tag{A1.11}$$

Q2. Why is the universe big?

Q3. Calculate the power spectrum of curvature perturbations $P_{\mathcal{R}_c}(k)$ produced during inflation driven by a potential $V = V_0\left(1 - 3\phi^2 + \ldots\right)$.

A3. The potential is of the form of Eq. (3.2.147) with

$$\frac{m^2}{V_0} = 6 \tag{A3.1}$$

and so for $\phi \ll 1$

$$a = e^{Ht} \tag{A3.2}$$

with

$$3H^2 \simeq V_0 \tag{A3.3}$$

and from Eqs. (3.2.149) and (3.2.150)

$$\phi = \phi_0 a^3 \tag{A3.4}$$

Therefore

$$\frac{a\dot{\phi}}{H} = 3\phi_0 a^4 \tag{A3.5}$$

Now

$$\eta = \int \frac{dt}{a} = -\frac{1}{aH} \tag{A3.6}$$

therefore

$$\frac{a\dot{\phi}}{H} = \frac{3\phi_0}{H^4\eta^4} \tag{A3.7}$$

and

$$\frac{H}{a\dot{\phi}}\left(\frac{a\dot{\phi}}{H}\right)'' = \frac{20}{\eta^2} \tag{A3.8}$$

Therefore Eq. (203) becomes

$$\varphi_k'' + k^2\varphi_k - \frac{20}{\eta^2}\varphi_k = 0 \tag{A3.9}$$

which from Eqs. (182) and (185) has solution

$$\varphi_k = i\sqrt{\frac{\pi}{4k}}\sqrt{-k\eta}\, H^{(1)}_{9/2}(-k\eta) \tag{A3.10}$$

$$\xrightarrow{k\eta\to 0} \frac{105}{\sqrt{2k^9}\,\eta^4} \tag{A3.11}$$

Matching this with Eq. (212)

$$\varphi_k \xrightarrow{k\eta\to 0} A_k \frac{a\dot{\phi}}{H} \tag{A3.12}$$

gives

$$A_k = \frac{105}{\sqrt{2k^9}\,\eta^4}\frac{H^4\eta^4}{3\phi_0} = \frac{35H^4}{\phi_0\sqrt{2k^9}} \tag{A3.13}$$

Therefore Eq. (220) gives

$$P_{\mathcal{R}_c}(k) = \frac{k^3}{2\pi^2}\left|A_k\right|^2 = \left(\frac{35H^4}{2\pi k^3\phi_0}\right)^2 = \left(\frac{35H}{2\pi\phi}\right)^2\Bigg|_{aH=k} \tag{A3.14}$$

Note that

$$\frac{\ddot{\phi}}{H\dot{\phi}} = 3 \not\ll 1 \tag{A3.15}$$

so the slow-roll formula

$$P_{\mathcal{R}_c}(k) \simeq \left(\frac{H^2}{2\pi\dot{\phi}}\right)^2\Bigg|_{aH=k} = \left(\frac{H}{6\pi\phi}\right)^2\Bigg|_{aH=k} \tag{A3.16}$$

is not accurate