Chapter 1

Differential Geometry

1.1 Tensors

A tensor is a mathematical object that directly represents a physical quantity. Tensors of the same type can be added, and multiplied by a scalar, in the usual way. Scalars and vectors are tensors, but many physical quantities are some other type of tensor.

A scalar is a tensor that behaves like a number. Examples of spatial\(^1\) scalars are time \(t\), energy \(E\) and electric potential \(\phi\). Examples of spacetime scalars are proper time \(\tau\), mass \(m\) and charge \(q\).

1.1.1 Vectors and covectors

A vector is a tensor that behaves like an arrow. Their properties inspire the vector space axioms of mathematics. A scalar times a vector is a vector and the sum of two vectors is a vector, see Figure 1.1.2. Examples of vectors are displacement \(\vec{d}x\), velocity

\[ \vec{d}x + \vec{v} = \vec{d}x + \vec{v} \]

Figure 1.1.2: The sum of two vectors is a vector.

\(^1\)Physical quantities may be one type of tensor with respect to one space but another type of tensor with respect to another space. For example, a displacement in time is a scalar with respect to space but a vector with respect to time. Unless otherwise specified, the space can be assumed to be space, or spacetime in the context of relativity.
\[ \vec{v} \equiv \frac{d\vec{x}}{dt} \] (1.1.1)

and acceleration

\[ \vec{a} \equiv \frac{d\vec{v}}{dt} \] (1.1.2)

A **covector** or **one-form** is a tensor that behaves like the local linearized form of contour lines or the crests of a wave, see Figure 1.1.3. A scalar times a covector is a covector and the sum of two covectors is a covector, see Figure 1.1.4. Examples of covectors are the gradient of a scalar field \( \nabla \phi \), wave “vectors” \( \vec{k} \), electric field and magnetic “vector” potential

\[ F = -\nabla \phi - \frac{\partial A}{\partial t} \] (1.1.3)

momentum

\[ p = \hbar \vec{k} \] (1.1.4)

and force

\[ F = qE \] (1.1.5)

or

\[ F = \frac{dp}{dt} \] (1.1.6)

A vector can be **contracted** with a covector to give a scalar

\[ \vec{v} \cdot \omega = \text{scalar} \] (1.1.7)

corresponding to the number of covector planes crossed by the vector, with sign given by the relative orientations of the vector and covector, see Figure 1.1.5. For example, a
displacement contracted with the gradient of a scalar field gives the change in the scalar field
\[ \dd x \cdot \nabla \phi = d\phi \] (1.1.8)
and power equals force contracted with velocity
\[ P = F \cdot \vec{v} \] (1.1.9)

Comparing vectors and covectors, the magnitude of a vector is given by its length, while the magnitude of a covector is given by the density of its planes. The direction of a vector is along its length (intrinsically oriented), while the direction of a covector is normal to its planes (extrinsically oriented) in the sense that \( \vec{n} \cdot \vec{v} = 0 \) for any vector \( \vec{v} \) lying in the plane of the covector \( \vec{n} \). Thus, a vector is an intrinsically oriented dimension one plane element, while a covector is an extrinsically oriented codimension\(^2\) one plane density.

### 1.1.2 Abstract index notation

The notation \( \vec{v} \) and \( \vec{\omega} \) works well for vectors and covectors but is inadequate for more general tensors. Instead, we will use the abstract index notation in which a vector \( \vec{v} \) is written \( v^a \) and a covector \( \vec{\omega} \) is written \( \omega_a \)

\[ \vec{v} \leftrightarrow v^a, \quad \vec{\omega} \leftrightarrow \omega_a \] (1.1.10)

The abstract index \( a \) denotes the tensorial nature of the quantity by its position and does not take specific values. A contraction is denoted by repeated indices

\[ \omega^a \cdot v^a \leftrightarrow \omega_a v^a \] (1.1.11)

A tensor can have an arbitrary number of vector and covector indices \( T^{ab\ldots} \).

A tensor \( T_{ab} \) can be decomposed into symmetric and antisymmetric parts

\[ T_{ab} = T_{(ab)} + T_{[ab]} \] (1.1.12)

where round brackets denote the symmetric part

\[ T_{(ab)} = \frac{1}{2} (T_{ab} + T_{ba}) \] (1.1.13)

and square brackets denote the antisymmetric part

\[ T_{[ab]} = \frac{1}{2} (T_{ab} - T_{ba}) \] (1.1.14)

\[ ^2 \text{Codimension } d \text{ is dimension } D - d \text{ where } D \text{ is the dimension of the space.} \]
1.1.3 Metric

The metric $g_{ab}$ and inverse metric $g^{ab}$ define lengths and angles in a space. They are symmetric tensors

$$g_{ab} = g_{ba}, \quad g^{ab} = g^{ba}$$

(1.1.15)

and related by

$$g^{ab} g_{bc} = \delta^a_c$$

(1.1.16)

where the identity tensor $\delta^a_b$ has the property

$$\delta^a_b v^b = v^a, \quad \delta^a_b \omega_b = \omega_a$$

(1.1.17)

and similarly for other tensors.

The metric gives the inner product of vectors and covectors

$$\vec{u} \cdot \vec{v} = g_{ab} u^a v^b, \quad \omega \cdot \sigma = g^{ab} \omega_a \sigma_b$$

(1.1.18)

Note that these inner products depend on the metric, in contrast to the contraction of a vector with a covector.

The metric converts vectors into covectors and vice versa

$$v_a = g_{ab} v^b, \quad v^a = g^{ab} v_b$$

$$\omega^a = g^{ab} \omega_b, \quad \omega_a = g_{ab} \omega^b$$

(1.1.19)

For example, the traditional vector representation $\vec{E}$ of the electric field $E$ is

$$E^a = g^{ab} E_b$$

(1.1.20)

More generally, the metric can raise or lower indices on any tensor \footnote{It is important to maintain the horizontal position of the indices if the tensor is not symmetric since $T^a_b = g_{bc} T^{ac} \neq g_{bc} T^{ca} = T^a_b$ if $T^{ac} \neq T^{ca}$.}

$$T^{a}{}_{b} = g_{bc} T^{ac}$$

(1.1.21)